Interval-Based Multicriteria Decision Making

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Abstract

The aim of this paper is to show how non-additive measures and intervals can be combined in order to provide a simple and accurate approach to multi-criteria decision making problems. We construct an interval-based Choquet integral in order to derive preferences over a set of multidimensional alternatives. Preferences are no longer real number comparisons, but interval comparisons, which is not straightforward to interpret. In this paper, we propose strategies of choice, and explain how we can integrate additional information – such as probabilities – to intervals, so as to ease the choice.

Key words: AI, multi-criteria decision making, interval computations, Choquet integral, preferences.

1 Introduction

In multicriteria decision making, we aim at ordering multidimensional alternatives and giving a semantic interpretation of the results. A traditional approach for the ordering problem is to use a weighted sum that ensures low complexity ($O(n)$) and ease of use and where each weight represents the (subjective) importance given by a decision maker to a particular attribute or criterion.

Despite its simplicity, it is difficult to deal with dependencies between criteria using additive approaches. To prevent this problem, non-additive measures and integrals can be used to represent preferences. An axiomatization of multicriteria decision making using the Choquet integral (a particular case of non-additive integral) was provided in [9].

We are interested in providing a practical solution for such problems using the Choquet integral. An inherent problem of non-additive measures is their exponential cost. However, the notion of 2-additive measures (see [5]) allows us to limit this cost to a $O(n^2)$. Besides, a convenient representation of the 2-additive Choquet integral allows us to express the Choquet integral in terms of complementary, redundant and independent criteria which is a natural extension of the weighted sum.

In practical problems, we only require the decision maker to provide importance and interaction indices which are sufficient to define preferences over the alternatives as long as we assume the measure to be 2-additive. However, it is unlikely that the decision maker can give precise values for these indices. Nevertheless, this is not a major problem as we can reasonably expect the decision maker to be able to give intervals of values.

Therefore, the aim of this paper is to present a Choquet integral based on intervals that allows us to express intervals of preferences for multidimensional alternatives. This will allow us to have a simple, yet accurate model of preferences.

First, we recall the essentials of MCDM and non-additive integration. Then, we present intervals and their operations, and describe how to combine these two theories to obtain interval of preferences in a MCDM setting. Finally, we present strategies of choice between intervals of preferences, and describe how to integrate probabilistic information in the intervals to free the choice from strategies.

2 Non-additive measures in MCDM

2.1 Multicriteria decision making

Let us define a multicriteria decision making problem as a triple $(X, I, (\succeq_i)_{i \in I})$ where $X \subset X_1 \times \cdots \times X_n$ is the set of alternatives of our problem and each set X_i is the set of values of attribute i. I is the (finite) set of criteria or attributes. And for all $i \in I$, $\succeq_i$ is a preference relation (a weak order) over X_i.

The problem is to "combine" the preference relations (or partial preferences) $\succeq_i$ in a rational way, in agreement with the decision maker's partial preferences.

2.2 Utility functions

A first step to achieve the construction of a global preference relation is to put all values of attributes over a common scale.

In this paper, we assume all the sets X_i to be order-separable. Then, we are guaranteed the existence of n monodimensional utility functions $u_i : X_i \to \mathbb{R}$ such that for all $x_i, y_i \in X_i$, $x_i \succeq y_i$ if and only if $u_i(x_i) \geqslant u_i(y_i)$ (see [7]).

In order to define a global preference $\succeq$ over X that is "consistent" with the partial orders, we now need to define a global utility function $u : X \to \mathbb{R}$ that aggregates the monodimensional utility functions u_i, that is, we need to build an aggregation operator $\mathcal{H} : \mathbb{R}n \to \mathbb{R}$ such that:

$$\forall x, y \in X \ , \ x \succeq y \Leftrightarrow \mathcal{H}(u_1(x_1), \cdots, u_n(x_n)) \geqslant \mathcal{H}(u_1(y_1), \cdots, u_n(y_n)) \quad (1)$$

with $x = (x_1, \cdots, x_n)$ and $u(x) = \mathcal{H}(u_1(x_1), \cdots, u_n(x_n))$.

In the sequel, we denote the global utility function by $u(x) = \mathcal{H}(u_1(x_1), \cdots u_n(x_n))$ for $x \in X$. By consistent, we mean that the choice of the aggregation operator should reflect the preferences of the decision maker.

A very natural and simple approach for such a problem is to use a simple weighted sum. The decision maker is asked to provide weights $\alpha_i \in [0, 1]$ that reflect the importance of each criterion and such that $\sum_{i=1}^{n} \alpha_i = 1$. The utility function is then defined by:

$$\forall x \in X \ , \ u(x) = \sum_{i=1}^{n} \alpha_i u_i(x_i) \quad (2)$$

However, we can show that using an additive aggregation operator such as a weighted sum is equivalent to assuming some kind of independence property (namely that all the attributes mutually preferentially independent ([8]).

This is not always desirable so, we need to turn to non-additive approaches.

The notion of mutual preferential independence is formally equivalent to the notion of the sure-thing principle in Decision under Uncertainty ([4]). The sure-thing principle also leads to paradoxes in Decision under Uncertainty and Schmeidler ([11], [12]) had proposed the use of non-additive measures and the Choquet integral as representation tools. We follow the same approach, and see how non-additive measures and the Choquet integral leads to more adequate representation of preferences in an MCDM context.

2.3 Non-additive measures and integrals

For the sake of our applications, we restrict ourselves to the finite case. However, these definitions can be extended to infinite sets (see [6] for a detailed presentation of fuzzy integration).

In the following definition, $\mathcal{P}(I)$ represents the power set of I.
Definition 1. *Let I be the set of attributes (or any set in a general setting). A set function $\mu : \mathcal{P}(I) \to [0, 1]$ is called a non-additive measure (or fuzzy measure) if it satisfies the three following axioms:*

(1) $\mu(\varnothing) = 0$: the empty set has no importance
(2) $\mu(I) = 1$: the maximal set has maximal importance

(3) $\mu(B) \leqslant \mu(C)$ if $B,\ C \subset I$ and $B \subset C$: a new criterion added cannot make the importance of a coalition (a set of criteria) diminish.

As the values of the empty set and of the maximal set are fixed, we need $2^n - 2$ values or coefficients to define a non-additive measure. So, there is a trade-off between complexity and accuracy. We will see later (Subsection 2.4) that we can reduce the complexity in order to guarantee that non-additive measures are used in practical applications.

Definition 2. *Let μ be a non-additive measure on $(I, \mathcal{P}(I))$ and an application $f : I \to \mathbb{R}+$. The Choquet integral of f w.r.t μ is defined by:*

$$(C)\int_I f d\mu = \sum_{i=1}^{n} (f(\sigma(i)) - f(\sigma(i-1)))\mu(A_{(i)}).$$

where σ is a permutation of the indices in order to have $f(\sigma(1)) \leqslant \cdots \leqslant f(\sigma(n))$, $A_{(i)} = \{\sigma(i), \ldots, \sigma(n)\}$ and $f(\sigma(0)) = 0$, by convention.

When there is no risk of confusion, we will write (i) for $\sigma(i)$. A Choquet integral ([3]) is a sort of weighted mean taking into account the importance of every coalition of criteria and is an extension of the Lebesgue integral.

2.4 Representation of preferences

We are now able to present how non-additive measures can be used in lieu of the weighted sum and other more traditional aggregation operators in a multicriteria decision making framework.

It was shown in [9] that under rather general assumptions over the set of alternatives X, and over the weak orders $\succeq_i$, there exists a unique non-additive measure μ over I such that:

$$\forall x, y \in X\ ,\ x \succeq y \Leftrightarrow u(x) \geqslant u(y) \tag{3}$$

where:

$$u(x) = \sum_{i=1}^{n} [u_{(i)}(x_{(i)}) - u_{(i-1)}(x_{(i-1)})]\mu(A_{(i)}) \tag{4}$$

which is simply the aggregation of the monodimensional utility functions using the Choquet integral w.r.t. μ.

We are still facing two crucial problems. First, the proof of the above result is not constructive. Second, as we have said before, evaluating a non-additive measure requires $O(2^n)$ values. We are going to see that we can overcome these difficulties and that using non-additive measures (coupled with intervals) offers a nice solution to multi-criteria decision making problems.

Let us start with a couple of definitions that will allow us to show how to limit the complexity to a $O(n^2)$.

The global importance of a criterion is given by evaluating what this criterion brings to every coalition it does not belong to, and averaging this input. This is given by the Shapley value or index of importance (see [5]).

Definition 3. *Let μ be a non-additive measure over I. The Shapley value of index j is defined by:*

$$v(j) = \sum_{B \subset I \setminus \{j\}} \gamma_I(B)[\mu(B \cup \{j\}) - \mu(B)]$$

with $\gamma_I(B) = \frac{(|I|-|B|-1)! \cdot |B|!}{|I|!}$, $|B|$ *denotes the cardinal of* B.

The Shapley value can be extended to degree two, to define the indices of interactions (between attributes). Indeed, as we will see below, the Shapley value defines the importance of an attributes, whereas the interaction indices define the level of interaction between attributes, which is interesting to give a semantic interpretation of the Choquet integral.

Definition 4. *Let μ be a non-additive measure over I. The interaction index between i and j is defined by:*

$$I(i,j) = \sum_{B \subset I - \{i,j\}} \xi_I(B) \cdot (\mu(B \cup \{i,j\}) - \mu(B \cup \{i\}) - \mu(B \cup \{j\}) + \mu(B))$$

with $\xi_I(B) = \frac{(|I|-|B|-2)! \cdot |B|!}{(|I|-1)!}$.

The interaction indices belong to the interval $[-1, +1]$ and

- $I(i,j) > 0$ if the attributes i and j are complementary;
- $I(i,j) < 0$ if the attributes i and j are redundant;
- $I(i,j) = 0$ if the attributes i and j are independent.

Interactions of higher orders can also be defined, however we will restrict ourselves to second order interactions which offer a good trade-off between accuracy and complexity. Indeed, we are interested in defining an approach that is both accurate and tractable. Defining the importance of attributes and the interaction between attributes is generally enough in MCDM problems, and restricting ourselves to interaction between two attributes guarantees us to be at most quadratic with respect to the number of attributes. To do so, we define the notion of 2-additive measure.

Definition 5. *A non-additive measure μ is called 2-additive if all its interaction indices of order equal or larger than 3 are null and at least one interaction index of degree two is not null.*

In this particular case of 2-additive measures, we can show ([5]) that the Shapley value and the interaction indices (of order two and higher) offer us an other way to represent a Choquet integral, as follows:

Theorem 1. *Let μ be a 2-additive measure. Then the Choquet integral can be* *computed by:*

$$(C)\int_I f d\mu = \sum_{I_{ij}>0} (f(i) \wedge f(j)) I_{ij} + \sum_{I_{ij}<0} (f(i) \vee f(j)) |I_{ij}| + \sum_{i=1}^{n} f(i)(I_i - \frac{1}{2}\sum_{j \neq i} |I_{ij}|). \quad (5)$$

Note that this expression justifies the above interpretation of interaction indices, as a positive interaction index corresponds to a conjunction (complementary) and a negative interaction index corresponds to a disjunction (redundant).

In the weighted sum case, we assume that the decision maker can provide us with the weights she/he puts on each criterion. However, we know that this model is inaccurate when trying to deal with dependencies. We could use a Choquet integral instead, as we have seen that they are a convenient and precise tool to model preferences. However, the complexity is very high. Therefore, in order to combine the best of the two worlds, we can ask the decision maker to give the Shapley values, as well as the interaction indices, and then use the reconstruction theorem 1 to obtain the aggregation operator, which is a Choquet integral w.r.t. to a 2-additive measure. Of course, we have to assume the measure to be 2-additive to use theorem 1. However, this is not a serious limitation as the importance and the 2-order interaction are enough to give a thorough semantic interpretation of the results.

Nevertheless, such an approach raises an other problem. How can we expect the decision maker to give a precise value for the importance and interaction indices? In order to overcome this hurdle, we introduce the concept of interval and see how it can be used efficiently to derive "interval of preferences".

3 Intervals

3.1 Interval Arithmetic

Interval Arithmetic is an arithmetic over sets of real numbers called *intervals*. Interval arithmetic has been proposed by Ramon E. Moore [10] in the late sixties in order to model uncertainty, and to tackle rounding errors of numerical computations. For a complete presentation of interval arithmetic, we refer the reader to [1].

Definition 6 (Interval). *An* interval *is a closed and connected set of real numbers. The set of intervals*[1] *is denoted by* $\mathbb{IR}$. *Every* $\boldsymbol{x} \in \mathbb{IR}$ *is denoted by* $[\underline{x}, \overline{x}]$, *where its bounds are defined by* $\underline{x} = \inf \boldsymbol{x}$ *and* $\overline{x} = \sup \boldsymbol{x}$.
For every $a \in \mathbb{R}$, *the interval point* $[a, a]$ *is also denoted by* a.

[1] Note that, in order to represent the real line with closed sets, $\mathbb{R}$ is compactified in the obvious way with the infinities $\{-\infty, +\infty\}$. The usual conventions apply: $(+\infty) + (+\infty) = +\infty$, *etc.*

Given a subset ρ of $\mathbb{R}$, the *convex hull* of ρ is the interval $\texttt{Hull}(\rho) = [\inf \rho, \sup \rho]$. The *width* of an interval $\boldsymbol{x}$ is the real number $w(\boldsymbol{x}) = \overline{x} - \underline{x}$. Given two real intervals $\boldsymbol{x}$ and $\boldsymbol{y}$, $\boldsymbol{x}$ is said to be *tighter than* $\boldsymbol{y}$ if $w(\boldsymbol{x}) \leqslant w(\boldsymbol{y})$.

Interval Arithmetic operations are set theoretic extensions of the corresponding real operations. Given $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{IR}$ and an operation $\diamond \in \{+, -, \times, \div\}$, we have: $\boldsymbol{x} \diamond \boldsymbol{y} = \mathsf{Hull}\{x \diamond y \mid (x, y) \in \boldsymbol{x} \times \boldsymbol{y}\}$.

Due to properties of monotonicity, these operations can be implemented by real computations over the bounds of intervals. Given two intervals $\boldsymbol{x} = [a, b]$ and $\boldsymbol{y} = [c, d]$, we have for instance: $\boldsymbol{x} + \boldsymbol{y} = [a + c, b + d]$. The associative law and the commutative law are preserved over $\mathbb{IR}$. However, the distributive law does not hold. In general, only a weaker law is verified, called semi-distributivity. We observe in particular that equivalent expressions over the real numbers are no longer equivalent when handling intervals: different symbolic expressions may lead to different interval evaluations.

For instance, consider the following three expressions equivalent over the real numbers: $x^2 - x$, $x(x-1)$ and $(x - \frac{1}{2})^2 - \frac{1}{4}$. When evaluated over the intervals, with $\boldsymbol{x} = [0, 1]$, we obtain the following results: $\boldsymbol{x}^2 - \boldsymbol{x} = [-1, 1]$, $\boldsymbol{x}(\boldsymbol{x} - 1) = [-1, 0]$, $(\boldsymbol{x} - \frac{1}{2})^2 - \frac{1}{4} = [-\frac{1}{4}, 0]$. Expressions formerly equivalent over the real numbers are not necessarily equivalent when extended to the intervals. This problem is known as the dependency problem of interval arithmetic, and a fundamental problem in interval arithmetic consists in finding expressions that lead to tight interval computations.

3.2 Interval Extensions

Interval arithmetic is particularly appropriate to represent outer approximations of real quantities. The range of a real function f over a domain D, denoted by $\boldsymbol{f}^u(D)$, can be computed by interval extensions.

Definition 7 (Interval extension). *An* interval extension *of a real function* $f : D_f \subset \mathbb{R}^n \rightarrow \mathbb{R}$ *is a function* $\boldsymbol{\varphi} : \mathbb{IR}^n \rightarrow \mathbb{IR}$ *such that*

$$\forall \mathbf{X} \in \mathbb{IR}^n,\ (\mathbf{X} \in D_f \ \Rightarrow\ \boldsymbol{f}^u(\mathbf{X}) = \{f(\boldsymbol{x}) \mid \boldsymbol{x} \in \mathbf{X}\} \subseteq \boldsymbol{\varphi}(\mathbf{X}))\,.$$

This inclusion formula is called Fundamental Theorem of Interval Arithmetic*. Interval extensions are also called* interval forms *or* inclusion functions*.*

This definition implies the existence of infinitely many interval extensions of a given real function. In particular, the weakest and tightest extensions are respectively defined by: $\mathbf{X} \mapsto [-\infty, +\infty]$ and $\mathbf{X} \mapsto \mathsf{Hull}\boldsymbol{f}^u(\mathbf{X})$.

The most common extension is known as the *natural extension*. Natural extensions are obtained from the expressions of real functions, and are *inclusion monotonic* [2], which means that given a real function f, its natural extension,

[2] This property follows from the monotonicity of interval operations

denoted $\boldsymbol{f}$, and two intervals $\boldsymbol{x}$ and $\boldsymbol{y}$ such that $\boldsymbol{x} \subset \boldsymbol{y}$, then $\boldsymbol{f}(\boldsymbol{x}) \subset \boldsymbol{f}(\boldsymbol{y})$. Since natural extensions are defined by the syntax of real expressions, two equivalent expressions of a given real function f generally lead to different natural interval extensions. In Figure 1, we see that both interval functions define interval extensions of f. However, one function is clearly better.

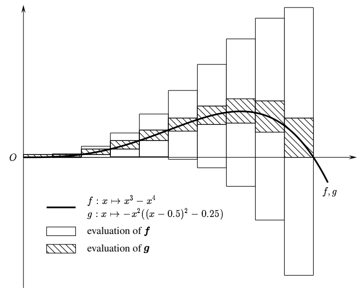

Fig. 1. Natural interval evaluations of two expressions of a real function f.

The overestimation problem, known as *dependency problem of IA*, is due to the decorrelation of the occurrences of a variable during interval evaluation. For instance, given $\boldsymbol{x} = [a, b]$ with $a \neq b$, we have: $\boldsymbol{x} - \boldsymbol{x} = [a - b, b - a] \supsetneq 0$. An important result is Moore's theorem known as the *theorem of single occurrences.*

Theorem 2 (Moore [10]). *Let t be a Σ-term and let the real function $f : D_f \subset \mathbb{R}^n \to \mathbb{R}$, $(x_1, \ldots, x_n) \mapsto t(x_1, \ldots, x_n)$ be the interpretation of t. If each x_i occurs only once in t, $1 \leqslant i \leqslant n$, then* $\forall \mathbf{X} \in \mathbb{IR}^n, (\mathbf{X} \subseteq D_f \;\Rightarrow\; \boldsymbol{f}^u(\mathbf{X}) = \boldsymbol{f}(\mathbf{X}))$.

In other words, there is no overestimation if all variables occur only once in the given expression.

4 Intervals of preferences

As we have seen before, to define preferences over alternatives, the user is required to provide importance and interaction indices, but is more likely to establish intervals of values than precise values. In this section, we explain how such interval information can be integrated in the scheme of computation of the Choquet integral, by extending its definition to Interval Arithmetic.

Since the user is not longer asked for precise values of indices I_{ij} and I_i, but for intervals [3], we consider intervals of values of these indices, and we respectively denote them $\boldsymbol{I_{ij}}$ and $\boldsymbol{I_i}$, i, $j \in \{1, \ldots, n\}$. As a consequence, the formula for the computation of the Choquet integral is now given by:

$$(C_{\mathbb{I}}) \textstyle\int_I f d\mu = \sum_{\boldsymbol{I_{ij}}>0} (f(i) \wedge f(j)) \boldsymbol{I_{ij}} \\ + \sum_{\boldsymbol{I_{ij}}<0} (f(i) \vee f(j)) |\boldsymbol{I_{ij}}| + \sum_{i=1}^{n} f(i) (\boldsymbol{I_i} - \tfrac{1}{2} \sum_{j \neq i} |\boldsymbol{I_{ij}}|). \quad (6)$$

[3] We will make the assumption (not restrictive) that the decision maker cannot give an interval whose interior contains 0, which would be a contradictory information.

$(C_{\mathbb{I}})$ means that the interpretation of this formula is performed using IA. As a consequence, the value of the integral is an interval, which we hope is the tightest one regarding the interval information provided by the user.

However, using IA means that overestimation of the range of real functions may occur, due to the above-mentioned dependency problem of IA. In particular, in the case of Equation 6, every interval variable $\boldsymbol{I_{ij}}$ occurs twice, with different monotonicities (once positively, once negatively), which inevitably leads to overestimating the expected range of values. Therefore, the right part of the formula is rewritten so as to obtain single occurrences only:

$$(C_{\mathbb{I}}) \int_I f d\mu = \sum_{\boldsymbol{I_{ij}}>0} \Big((f(i) \wedge f(j)) - \tfrac{1}{2}(f(i) + f(j)) \Big) \boldsymbol{I_{ij}} \\ + \sum_{\boldsymbol{I_{ij}}<0} \Big((f(i) \vee f(j)) - \tfrac{1}{2}(f(i) + f(j)) \Big) |\boldsymbol{I_{ij}}| \; + \; \sum_{i=1}^{n} f(i) \boldsymbol{I_i}$$

This formula contains only single occurrences of interval variables, which is a guarantee to obtain the exact range of possible values, given the intervals of preferences of the user.

4.1 Traditional against interval Choquet integral

It is worth noting that if the decision maker gives precise values for the importance and interaction indices (*i.e.,* real values), then the interval-based Choquet integral restricts theoretically to a standard Choquet integral and the intervals of preferences are real valued numbers. In practice however (i.e., when using a computer), when evaluating the *interval* Choquet integral on "real numbers", the evaluation turns out to be still an interval. Yet, this is an interesting feature since:

- if all intermediate values of the computation of the Choquet integral are floating-point numbers, the interval result will be an interval $[a, a]$, where a is the floating-point value of the Choquet integral;
- and otherwise, the computation of the interval Choquet integral results in an interval $[a, b]$, where a and b are floating-points, $a \leqslant c \leqslant b$, and c is the actual (expected) value of the Choquet integral.

In both cases, we retrieve the actual value of the Choquet integral (exactly or included).

4.2 Making decisions: an issue when using intervals

Two alternatives are compared w.r.t. the corresponding interval values of their interval integral of Choquet. Unfortunately, intervals may not be as easy to compare as real numbers.

The ideal case is the following:

$$(C_{\mathbb{I}}) \int_I f d\mu \succeq (C_{\mathbb{I}}) \int_I g d\mu \overset{def}{\Leftrightarrow} \underline{(C_{\mathbb{I}}) \int_I f d\mu} \geqslant \overline{(C_{\mathbb{I}}) \int_I g d\mu}$$

when the intervals of preferences do not intersect. In this case, alternative f is evaluated with values that are all better than those of alternative g. Here the preference is clear, and is interpreted as the alternative f is preferred to the alternative g.

However, the above case is very specific and may unfortunately not always happen. Indeed it may rather happen that:

$$(C_{\mathbb{I}}) \int_I f d\mu \bigcap (C_{\mathbb{I}}) \int_I g d\mu \neq \varnothing$$

In such a case, we need to define a degree of preference corresponding to the intersection of the intervals. Next section describes strategies to make decisions in such cases.

5 Strategies of preference

5.1 Naive strategy

A trivial solution could consist in having a look :

- either at the upper bounds and give preference to the highest upper bound, which corresponds to an optimistic behavior: the preference is given to the alternative more likely to have a high Choquet integral value;
- at the lower bounds and give preference the the highest lowest bound which then corresponds to a pessimistic behavior: the preference is given to the alternative less likely to have a low Choquet integral value.

However, many alternatives between the very optimistic case and the very pessimistic case are possible. It is our feeling that we need to look simultaneously at the upper and lower bounds as well as the width of the intervals. Indeed, in many situations, the decision maker will exhibit some sort of aversion of risk and will want to have intervals as tight as possible, that is restrict the degree of uncertainty. In particular, we can already draw some strategies of choice as follows.

5.2 Intermediate strategies

Suppose that we consider two intervals I and J, corresponding respectively to:

$$(C_{\mathbb{I}}) \int_I f d\mu \;\text{ and }\; (C_{\mathbb{I}}) \int_I g d\mu.$$

If the configuration is such that $\overline{I} > \overline{J}$ and $\underline{I} > \underline{J}$, then an optimistic strategy could consist in preferring to interval I, since I offers the possibility of having higher Choquet integral values. It is not as simple when J is included in I.

• Indeed, when the configuration is such that $\underline{J} - \underline{I} = \overline{I} - \overline{J}$, without more information, we can guess that there is the same probability for values in I to be smaller than values in J, as to be greater. As a consequence, a reasonable

strategy could consist in giving preference to J since J is tighter and therefore more accurate.
One the other hand, a risky but defensible strategy would consist in preferring I, in the hope of getting better values, *i.e.,* those greater than J's.

• When interval I is not as well-balanced around J as it was in the previous configuration, two configurations are to be considered. In such cases, our feeling is that we may have to give preference to the interval that minimizes the risk of having small Choquet integral values.

⋆ The first case is defined by: $\underline{J} - \underline{I} \ > \ \overline{I} - \overline{J}$. A safe strategy may consist in preferring J for which the probability of obtaining small Choquet integral values is less than for I.
⋆ On the contrary, the second case is defined by: $\underline{J} - \underline{I} \ < \ \overline{I} - \overline{J}$, and a safe strategy would then consist in preferring I for the same reason as just mentioned.

5.3 Degree of preference and corresponding strategies

It is our feeling that we need to look simultaneously at the upper and lower bounds as well as the width of the intervals. Indeed, in many situations, the decision maker will exhibit some sort of aversion of risk and will want to have intervals as tight as possible, that is restrict the degree of uncertainty.

In this respect, we define hereafter a degree of preference $d(I, J)$, which is intended to express the extent to which a better value of the Choquet integral is likely to lie in interval I, instead of in interval J. It is defined as follows: $d: \ \mathbb{I}^2 \ \longrightarrow [0, 1]$, where:

$$d(I,J) = \begin{cases} \frac{\overline{I}-\overline{J}}{|(\overline{I}-\overline{J})+(\underline{J}-\underline{I})|} & \text{if } \overline{I} > \overline{J} \text{ and } \underline{J} \geqslant \underline{I} \\ 1 & \text{if } \overline{I} = \overline{J} \text{ and } \underline{I} > \underline{J} \\ & \text{or: if } \overline{I} > \overline{J} \text{ and } \underline{I} \geqslant \underline{J} \\ & \text{or: if } \overline{I} = \overline{J} \text{ and } \underline{I} = \underline{J} \\ 1 - d(J,I) & \text{otherwise} \end{cases} \tag{7}$$

Using this definition, we evaluate the chances to get a better value in interval I by comparing the "positive" [4] width of interval $[\overline{J}, \overline{I}]$, *i.e.,* the interval where a better value of I could lie, to that of interval $[\underline{I}, \underline{J}]$, *i.e.,* the interval where a worse value of I could lie.

5.3.1 Strategies associated to degrees of preference.

Any strategy can then be defined by a degree, above which the interval with the largest upper bound is preferred to the other. In particular, we can categorize

[4] By positive, we mean that if $\overline{J} > \overline{I}$, then interval $[\overline{J}, \overline{I}]$ is not an interval (bounds inverted) and its width is going to be $-w([\overline{I}, \overline{J}])$

strategies as follows:

- a risky strategy will be associated with a low degree d: this is an optimistic scenario in which, even if it is more likely [5] to get a worse value in interval I than in J, the strategy focuses on the slight, yet existing, possibility of getting a better value in I;
- a safe strategy will be associated with a high degree d: this is a pessimistic scenario where below degree d, even if there are chances to get better values in I, the focus is made on the remaining possibilities that this case does not happen, *i.e.,* that we get worse values in I.

As a result, we will always be able to compare two strategies, in term of safeness or riskiness, regarding their associated degree.

5.3.2 Comparison of intervals.

The above-described degree of preference allows us to compare intervals resulting from the computation of interval Choquet integrals. In particular, given two intervals I and J, with $\overline{I} \geqslant \overline{J}$, and a strategy s associated with a degree $d \in [0,1]$, interval I is preferred to interval J if and only if $d(I,J) \geqslant d$:

$$I \succeq J \quad \overset{def.}{\Leftrightarrow} \quad \begin{cases} d(I,J) \geqslant d \\ \overline{I} \geqslant \overline{J} \end{cases} \quad \overset{def.}{\Leftrightarrow} \quad J \not\succeq I$$

(otherwise we note $J \succeq I$)

The comparison relies indeed on the capacity of the interval of largest upper bound to provide a better value.

Proposition 1. *Relation $\succeq$ implies a partial order on the set $\mathbb{I}$ of intervals.*

Proof. A complete proof can be found in [2]. □

5.3.3 Need for generalization.

We must point out that this evaluation of the degree relies on the fact that any values are equally possible in the compared intervals. However, this does not necessarily hold: situations may arise in which the user has more knowledge about the indices than only intervals of values, *i.e.,* bounds. Indeed, the user may have a stronger intuition about the part of the interval where the actual value may lie. This knowledge may be of different kinds. In particular, we address, in [2], the case in which the user is able to provide probabilistic distributions in addition to intervals of values.

6 Conclusion

In this paper, we have presented a simple computation scheme, combining the Choquet integral (in the 2-additive case) with interval arithmetic that allows us to give intervals of preferences over multidimensional alternatives.

[5] The likeliness is based upon the width of intervals only.

The approach is very attractive as it reflects more accurately what we can really expect from a decision maker, yet remains simple and still allows us to represent dependencies between attributes which is not possible with more traditional approaches such as the weighted sum.

In the case where the intervals of preferences are disjoint, the order of alternatives is clearly established. However, it is not as trivial in the (more probable) case where the intervals have an intersection. In this case, strategies have been presented, as well as a general degree of preference.

Future work will consist in extending our interval-decision framework to other kinds of knowledge provided by the user, such as general distributions.

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