Interval-based Multi-Criteria Decision Making: Strategies to Order Intervals

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***Abstract*—Ordering alternatives in interval-based multi-criteria decision making problems is not a trivial task when the intervals of preference are overlapping. In this paper, we aim at giving a rational and natural way of ranking alternatives by computing the degrees of preference, taking into consideration the upper and lower bounds of the interval of preference as well as its width. We slightly modify the general description of degree of preference to accommodate the strategy of choice for risk-prone and risk-averse individuals as well as situations where more information is available (e.g., a probability distribution over the intervals).**

I. Introduction

In multi-criteria decision making (MCDM), the goal is to order multi-dimensional alternatives so that the order is consistent with the decision maker's preferences. The traditional approaches to aggregate all dimensions of an alternative into a single value, which are based on additive methods, fail in general, as they have to rely on independence properties (e.g., mutual preferential independence [8]) that are usually not satisfied. The authors have proposed an extension based on the Choquet integral (a non-addtive integral) with respect to a 2-additive measure. The decision maker is asked to provide values of importance for each criterion as well as interaction indices of degree 2 and the Choquet integral is computed from these values using a reconstruction formula provided in [5].

Nevertheless, this approach faces a common hurdle: the decision maker is asked to provide subjective and precise values of importance of each criterion and interaction indices among criteria. Practically, it is unlikely that the decision maker can give precise values for these indices. However, we can reasonably expect the decision maker to be able to give ranges of required values.

Intervals are a natural approach to solve the issue of imprecise weights given by the decision-maker to different criteria. However, they bring a new difficulty: comparing intervals may not be as straightforward as comparing real numbers, so strategies to order intervals must be given.

Another issue we face when ordering intervals is that we would like to be able to represent a risk-prone or a risk-averse behavior. Intuitively, a risk-averse decision maker would be more interested in the lower values of the interval whereas a risk taker will be willing to consider only the higher values, representing an increased risk, for an increased profit. We expect a decision maker to have a stand of how much risk he/she is willing to take, and again, it is more reasonable to expect that the level of risk be given in terms of an interval rather than a single number.

The aim of this paper is to present general strategies to order intervals, which result from calculating interval-valued Choquet integrals to represente the global values of multi-dimensional alternatives.

In the first part of this paper, we recall the essentials of MCDM and non-additive integration, mostly in the discrete case, basics of intervals, and how to combine these theories to obtain interval of preferences in a MCDM setting. Then, we present strategies of choice between intervals of preferences, and we describe how to integrate other available information, such as the level of risk the decision maker is willing to accept and probabilistic information, in the decision. Finally, we present a simple application that uses the tools presented in the paper to reach the best solution.

II. Multi-criteria Decision Making

A multicriteria decision making problem could be defined as a triple $(X, I, (\succeq_i)_{i\in I})$ where

- $X \subset X_1 \times \cdots \times X_n$ is the set of alternatives with each set X_i representing a set of values of the attribute i.
- I is the (finite) set of criteria (or attributes).
- $\forall i \in I$, $\succeq_i$ is a preference relation (a weak order) over X_i.

The next task is to "combine" the preference relations $\succeq_i$ in a rational way, in agreement with the decision maker's partial preferences.

The natural way to construct a global preference is by using utility function for each attribute to reflect partial preferences of a decision-maker, and then combine these monodimensional utilities into a global utility function using an aggregation operator. The existence of monodimensional utility functions is guaranteed under relatively loose hypotheses by the work presented in ([8]).

The simplest method for combining monodimensional utilities is a weighted sum approach, in which the decision maker is asked to provide weights that reflect the importance of each criterion. Even though this approach is attractive due to its low complexity, it can be shown that using an additive aggregation operator, such as weighted sum, is equivalent to assuming that all the attributes are independent ([9]). In practice, this is not realistic and therefore, we need to turn to non-additive approaches, that is to aggregation operators that are not linear combinations of partial preferences.

Definition 1. (Non-additive measure) *Let I be the set of attributes and $\mathcal{P}(I)$ the power set of I. A set function $\mu : \mathcal{P}(I) \rightarrow [0,1]$ is called a non-additive measure (or fuzzy measure) if it satisfies the following three axioms:*

(1) $\mu(\emptyset) = 0$ *: the empty set has no importance.*
(2) $\mu(I) = 1$ *: the maximal set has maximal importance.*
(3) $\mu(B) \leq \mu(C)$ *if $B,\ C \subset I$ and $B \subset C$: a new criterion added cannot make the importance of a coalition (a set of criteria) diminish.*

Of course, any probability measure is also a non-additive measure. Therefore non-additive measure theory is an extension of traditional measure theory. Moreover, a notion of integral can also be defined over such measures.

A non-additive integral, such as the Choquet integral ([3]), is a type of a general averaging operator that can model the behavior of a decision maker. The decision-maker provides a set of values of importance, this set being the values of the non-additive measure on which the non-additive integral is computed from.

Formally, The Choquet integral is defined as follows:

Definition 2. (Choquet integral) *Let μ be a non-additive measure on $(I, \mathcal{P}(I))$ and an application $f : I \rightarrow \mathbb{R}^+$. The Choquet integral of f w.r.t. μ is defined by:*

$$(C) \int_I f d\mu = \sum_{i=1}^{n} (f(\sigma(i)) - f(\sigma(i-1)))\mu(A_{(i)}),$$

where σ is a permutation of the indices in order to have $f(\sigma(1)) \leq \cdots \leq f(\sigma(n))$, $A_{(i)} = \{\sigma(i), \ldots, \sigma(n)\}$, and $f(\sigma(0)) = 0$, by convention.

It can be shown that many aggregation operators can be represented by Choquet integrals with respect to some fuzzy measure. Note that there are other non-additive approaches to decision making besides the Choquet integral, one of them being the Sugeno integral ([12]):

Definition 3. (Sugeno integral) *Let μ be a fuzzy measure on $(I, \mathcal{P}(I))$ and an application $f : I \rightarrow [0, +\infty]$. The Sugeno integral of f w.r.t. μ is defined by:*

$$(S) \int f \circ \mu = \bigvee_{i=1}^{n} (f(x_{(i)}) \wedge \mu(A_{(i)})).$$

where $\vee$ is the supremum and $\wedge$ is the infimum.

Even though the Choquet and the Sugeno integrals are structurally similar, their applications are very different. The Choquet integral is generally used in quantitative measurements, while Sugeno integral has found more applications in qualitative approaches. For the purpose of this paper, we restrict ourselves to quantitative approaches.

Although the Choquet integral is well suited for quantitative measurements, it has a major drawback. We need a decision maker to input a value of importance of each subset of attributes, which leads to an exponential complexity and is therefore intractable. We can overcome intractability by using 2-additive measure to limit the complexity to a $O(n^2)$ (as shown in [2]).

Before giving the definition of 2-additive measure, we need to define notion of interaction indices of orders 1 and 2. The importance of an attribute (or the interaction index of degree 1) is best described as the value this attribute brings to each coalition it does not belong to. It is given by the Shapley value ([11]):

Definition 4. (Shapley value) *Let μ be a non-additive measure over I. The Shapley value of index j is defined by:*

$$v(j) = \sum_{B \subset I \setminus \{j\}} \gamma_I(B)[\mu(B \cup \{j\}) - \mu(B)]$$

with $\gamma_I(B) = \frac{(|I|-|B|-1)! \cdot |B|!}{|I|!}$ and $|B|$ denoting the cardinal of B.

While the Shapley value gives the importance of a single attribute to the entire set of attributes, the interaction index of degree 2 represents the interaction among two attributes, and is defined by ([4],[6]):

Definition 5. (Interaction index of degree 2) *Let μ be a non-additive measure over I. The interaction index between i and j is defined by:*

$$\begin{aligned} I(i,j) \quad &= \quad \textstyle\sum_{B \subset I \setminus \{i,j\}} (\ \xi_I(B) \cdot (\mu(B \cup \{i,j\}) \\ &\qquad -\mu(B \cup \{i\}) - \mu(B \cup \{j\}) + \mu(B)) \end{aligned}$$

with $\xi_I(B) = \frac{(|I|-|B|-2)! \cdot |B|!}{(|I|-1)!}$.

The interaction indices belong to the interval $[-1, +1]$ and

- $I(i,j) > 0$ if the attributes i and j are complementary;
- $I(i,j) < 0$ if the attributes i and j are redundant;
- $I(i,j) = 0$ if the attributes i and j are independent.

Even though we can define interaction indices of any order, defining the importance of attributes and the interaction indices between two attributes is generally enough in MCDM problems. Thus, 2-additive measures constitute a feasible and accurate tool in this setting. The formal definition of 2-additive measure follows ([4]):

Definition 6. (2-additive measure) *A non-additive measure μ is called 2-additive if all its interaction indices of order equal to or larger than 3 are null and at least one interaction index of degree two is not null.*

We can also show ([5]) that the Shapley values and the interaction indices of order two offer us an elegant way to represent a Choquet integral. Therefore, in a decision-making problem, we can ask the decision maker to give the Shapley values, I_i, and the interaction indices, I_{ij}, and then use the Choquet integral w.r.t. to a 2-additive measure, μ, to obtain the aggregation operator:

$$(C)\int_I f d\mu = \sum_{I_{ij}>0} (f(i) \wedge f(j)) I_{ij} + \sum_{I_{ij}<0} (f(i) \vee f(j)) |I_{ij}| + \sum_{i=1}^{n} f(i)(I_i - \frac{1}{2}\sum_{j\neq i} |I_{ij}|).$$

Nevertheless, this approach raises another problem. We cannot expect a decision maker to give precise values for the importances and interaction indices. In order to overcome this hurdle, it was shown ([2]) that the use of intervals provides a nice solution.

III. Intervals

Interval Arithmetic (IA) is an arithmetic over sets of real numbers called *intervals*. It has been studied since late fifties in order to model uncertainty and to tackle rounding errors of numerical computations. For a complete presentation of interval arithmetic, we refer the reader to ([1],[7]).

Definition 7. (Interval) *A closed real interval is a closed and connected set of real numbers. The set of closed real intervals is denoted by $\mathbb{IR}$. Every $\boldsymbol{x} \in \mathbb{IR}$ is denoted by $[\underline{x}, \overline{x}]$, where its bounds are defined by $\underline{x} = \inf \boldsymbol{x}$ and $\overline{x} = \sup \boldsymbol{x}$. For every $a \in \mathbb{R}$, the interval point $[a, a]$ is also denoted by a.*

In the following, elements of $\mathbb{IR}$ are simply called *real intervals* or *intervals*.

The *width* of a real interval $\boldsymbol{x}$ is the real number $w(\boldsymbol{x}) = \overline{x} - \underline{x}$. Given two real intervals $\boldsymbol{x}$ and $\boldsymbol{y}$, $\boldsymbol{x}$ is said to be *tighter than* $\boldsymbol{y}$ if $w(\boldsymbol{x}) \leq w(\boldsymbol{y})$.

A. Intervals of preferences

As mentioned earlier, to define preferences over multi-dimensional alternatives, the user is required to provide importance and interaction indices, but is more likely to provide intervals of values $\boldsymbol{I_i}$ and $\boldsymbol{I_{ij}}$, i, $j \in \{1, \ldots, n\}$, which leads to evaluation of a Choquet integral over intervals using IA.

However, an issue arises when using intervals to evaluate alternatives: intervals are not as easy to compare as real numbers.

IV. Strategies of preference

When comparing intervals, the ideal case is when the intervals of preferences do not intersect. In this case, if alternative I is evaluated with values that are all better than those of alternative J, the preference is clearly given to the alternative I.

However, the above case is very specific and unfortunately does not happen often. More common is that two intervals intersect and we need to choose a better of two overlapping intervals. The following sections describe strategies to make decisions in such cases.

A. Naive strategy

A straightforward solution consists in comparing only the upper bounds and giving the preference to the interval with highest upper bound (which corresponds to an optimistic behavior of a decision-maker as he/she is only interested in the highest potential values rather than all the values that could be reached) or comparing the lower bounds and giving preference to the highest lower bound (which corresponds to a pessimistic behavior).

However, many alternatives between the very optimistic case and the very pessimistic case are possible. They require us to look simultaneously at the upper and lower bounds as well as the width of the intervals, which highlights the degree of uncertainty of the alternative's value. To combine these variables, a degree of preference was introduced ([2]).

B. Degree of preference and strategy of preference

A degree of preference, $d(I, J)$, intended to express the extent to which a better value of the Choquet integral is likely to lie in interval I, rather than in interval J.

It is defined as a function $d: \mathbb{I}^2 \rightarrow [0, 1]$, where:

$$d(I,J) = \begin{cases} \frac{\overline{I}-\overline{J}}{|(\overline{I}-\overline{J})+(\underline{J}-\underline{I})|} & \text{if } \overline{I} > \overline{J} \text{ and } \underline{J} \geq \underline{I} \\ 1 & \text{if } \overline{I} = \overline{J} \text{ and } \underline{I} > \underline{J} \\ & \text{or: if } \overline{I} > \overline{J} \text{ and } \underline{I} \geq \underline{J} \\ & \text{or: if } \overline{I} = \overline{J} \text{ and } \underline{I} = \underline{J} \\ 1 - d(J,I) & \text{otherwise} \end{cases} \quad (1)$$

The higher the value of the degree of preference, the greater the chance that the optimal interval is the interval I, while lower value of the degree of preference implies that the interval J would more likely contain higher value of Choquet integral.

C. Level of Risk a Person is Willing to Take

The degree of preference, as described above, assumes that a decision-maker is risk-neutral, that is the person is not willing to undergo an extreme risk nor he/she believes that there is a reason to be too careful. However, sometimes, a person exhibits a risk-prone attitude and leans towards optimistic behavior, or on the other hand, the decision-maker could be more risk-averse especially if there is a reason to expect pessimistic results. If a decision-maker could provide the level of risk that he/she is willing to take in order to maximize the utility, then we can modify the degree of preference to include this fact.

Let us assume that the level of risk a person wants to take is expressed by a real value in the interval $[0, 1]$, where naturally, values close to 0 represent pessimistic situations, and values closer to 1 mean more optimistic expectations. Now, we can tighten the considered interval to better suit this level of risk. The shrinking of the interval $[\underline{X}, \overline{X}]$ based on the risk level $r \in [0, 1]$ is done in the following way:

- First, calculate the proportion of the interval that is considered important by the decision-maker:
$$p = 2 \cdot \min\{r - 0, 1 - r\}. \quad (2)$$
- Next, calculate the size of the interval that corresponds to the given proportion:
$$\text{size} = p \cdot (\overline{X} - \underline{X}). \quad (3)$$
- Finally, calculate the interval of importance, $[\underline{N}, \overline{N}]$:
$$[\underline{N}, \overline{N}] = \begin{cases} [\underline{X}, \underline{X} + size] & \text{if } r \leq 0.5 \\ [\overline{X} - size, \overline{X}] & \text{otherwise} \end{cases} \quad (4)$$

This approach clearly returns a single point instead of an interval in cases when the level of risk is at extreme points, *i.e.,* the interval of importance is the upper bound of the original interval when the risk level is 1, and the lower bound when the risk level is 0. In both cases, the problem is reduced to comparison of single (extreme) points rather than intervals, the situation that corresponds to the naive strategy.

Once we have tightened the intervals to reflect the level of risk the decision-maker is willing to take, we apply equation (1) to new intervals of importance to calculate the degree of preference, which determines better of two intervals.

Example 1. To demonstrate the advantage of calculating the degree of preference over intervals of importance rather than over the intervals that do not take into consideration the level of risk a decision-maker is willing to take, we look at a simple hypothetical example. Let us assume that the interval based Choquet integral resulted in the intervals $[\underline{I}, \overline{I}] = [0, 10]$ and $[\underline{J}, \overline{J}] = [6, 9]$ for two different alternatives, I and J, respectively. If a person making the decision is risk neutral, it is intuitive that this person would prefer alternative J for two reasons. First, the interval corresponding to alternative J is tighter than that of alternative I, which leads to less uncertainty. Second, the amount that alternative I could bring higher than alternative J is small in comparison to the lower values that alternative I could result in. Besides intuitive reasoning, we can show this by calculating the degree of preference:

$$d(I, J) = \frac{10 - 9}{|10 - 9| + |6 - 0|} = \frac{1}{7} \approx 0.143,$$

which tells us that interval corresponding to alternative I is less preferable than the interval related to alternative J.

However, if we consider a decision-maker who exhibits an optimistic behavior, say at a level of 0.9, intuitively this person does not worry about the low values that could possibly result from calculating interval Choquet integral, and thus, it might look that the better option is the alternative I since it could result in higher value of utility. We can show this by calculating the interval of importance for intervals of both alternatives, and then calculating the degree of preference over these intervals.

Thus, we first obtain the intervals of importance $N_I = [8, 10]$ and $N_J = [8.4, 9]$. Then, we calculate the degree of preference over the alternatives I and J using the intervals N_I and N_J:

$$d(I, J) = \frac{10 - 9}{|10 - 9| + |8.4 - 8|} = \frac{1}{1.4} \approx 0.714,$$

which clearly indicates the preference of alternative I.

With this example, we showed how the level of risk a person would accept to take can change the preferability of an interval over another. It also shows that the proposed method is in an agreement with the intuitive behavior.

D. Level of Risk Expressed as an Interval

The presented approach to determine the better of two intervals given the level of risk works well if the decision-maker can provide the exact degree of risk he/she is willing to take. However, in reality it is hard to describe the level of risk by a single number. More probable is that a person could define the level of risk by an interval $r = [\underline{r}, \overline{r}]$. In this case, the calculation of the interval of importance that encounters for the optimism/pessimism of a person is a bit more complicated. Instead of a precise interval, the result is an interval whose bounds are themselves intervals (2nd order interval), and therefore, the degree of preference would result in an interval, $d(I, J) = [\underline{d}, \overline{d}]$ rather than a single number. Three situations could occur:

- $\overline{d} < 0.5$ in which case the preferable choice is interval J.
- $\underline{d} > 0.5$ in which case the preferable choice is interval I.
- $0.5 \in [\underline{d}, \overline{d}]$ in which case the preferable choice is
  (1) interval I if $(\overline{d} - 0.5) \geq (0.5 - \underline{d})$
  (2) interval J otherwise.

To see the benefit of this approach, let us go back to Example 1., and consider an interval of the level of risk rather than a single number.

Example 2. Assume that the person could not give us a precise level of risk, but rather an interval, say [0.8,0.9]. Intuitively, it's not clear anymore what the person should do since there is still an optimistic behavior, but it's not high enough to easily determine the interval of preference. In this case, we follow the procedure to calculate the degree of preference for both upper and lower bounds of the interval of risk, and end up with two different degrees of preference. Using the lower bound of the risk level 0.8, the degree of preference is: $\underline{d} = \frac{10-9}{|10-9|+|7.8-6|} = \frac{1}{2.8} \approx 0.357$, while using the upper bound of the level of risk 0.9, we obtain $\overline{d} \approx 0.714$ as calculated in the first example. Thus, the degree of preference belongs to the interval $[0.357, 0.714]$. This interval shows that there is a higher probability that the exact value would be above 0.5. Thus, the preferable choice is the alternative I.

This example clearly illustrates that in some situations, it is not easy to just look at two intervals and determine the better one. It also explains how the level of risk a decision maker is willing to take can be taken into consideration to determine the best alternative.

E. Considering Different Probability Distributions

All of the above rankings of intervals suppose uniform probability distribution, which is a reasonable assumption if no additional information is available. However, sometimes more information is accessible and more accurate probability distribution over an interval could be considered. In this case, we can slightly modify the approach of calculating the degree of preference.

As before, we start by tightening the given interval based on the level of risk, r, that a person is willing to take. Thus, we need to determine the value, s, within the given interval $[\underline{X}, \overline{X}]$ such that

$$s = \begin{cases} P(x \leq 2r) & \text{if } r < 0.5 \\ P(x \geq (2r-1)) & \text{if } r > 0.5 \end{cases} \tag{5}$$

So the interval of importance is:

$$[\underline{N}, \overline{N}] = \begin{cases} [\underline{X}, \underline{X}+s] & \text{if } r \leq 0.5 \\ [\overline{X}-s, \overline{X}] & \text{otherwise} \end{cases} \tag{6}$$

Note that the above formula when applied to the uniform distribution leads exactly to the equation (4), with s replacing the variable *size*.

The next step is to calculate the degree of preference between two intervals given their new bounds. Encountering the probability distribution, the degree of preference is given by:

$$d(I,J) = \begin{cases} \frac{P(\overline{J} \leq x \leq \overline{I})}{P(\overline{J} \leq x \leq \overline{I}) + P(\underline{I} \leq x \leq \underline{J})} & \text{if } \overline{I} > \overline{J} \text{ and } \underline{J} \geq \underline{I} \\ 1 & \text{if } \overline{I} = \overline{J} \text{ and } \underline{I} > \underline{J} \\ & \text{or: if } \overline{I} > \overline{J} \text{ and } \underline{I} \geq \underline{J} \\ & \text{or: if } \overline{I} = \overline{J} \text{ and } \underline{I} = \underline{J} \\ 1 - d(J,I) & \text{otherwise} \end{cases} \tag{7}$$

When applied to uniform distribution, this equation simplifies to equation (1).

Remark. According to the approach we presented in this paper, a decision-maker is asked to give intervals of importance of each attribute and interaction indices between each two criteria. In many situations, if the width of interval is not limited, it is common that a decision-maker would give an interval bigger than what he/she really believes the interval should be to cover any possible extreme value even though the extreme values would very rarely happen. Thus, it is not uncommon that the values within an interval would not follow uniform distribution, but rather a form of Gaussian distribution (possibly screwed). In this situation, it is reasonable to assume that the interval Choquet integral would also not follow uniform distribution but would rather have higher probability of values in the interior of the interval than those close to bounds.

Example 3. Let us see how the knowledge of probability distribution could influence the ordering of alternatives. For this purpose, we revisit Example 1, with the assumption that the intervals resulting from Choquet integrals follow normal distribution. Also, assume that the interval for alternative I has mean $\mu_I = 5$ and standard deviation $\sigma_I = \frac{10}{3}$ so that almost 100% of the values fall within the interval $[0, 10]$. For same reason, assume that the interval corresponding to alternative J has mean $\mu_J = 7.5$ and standard deviation $\sigma_J = \frac{1}{2}$.

Following the normal distribution tables, the intervals of importance for the decision-maker's risk level of 0.9 are the following:

$$N_I = [6.4, 10] \text{ and } N_J = [7.92, 9].$$

To calculate the degree of preference, we use the characteristics of the distribution corresponding to alternative I since only the alternative I covers the possibility of having values in the intervals $[9, 10]$ and $[6.4, 7.92]$ The degree of preference is then equal to:

$$d = 0.129$$

which clearly indicated that the alternative J is better choice. This result might not be intuitive at first, but if we think for a moment and realize that the alternative I could bring better result only when value of utility is greater than 9, which is the tiny tail of a normal distribution, it is clear that this situation would not happen very often.

V. An Example of Application

In this section, we present a simplified example from the world of finance that benefits from the approach presented in the paper. Buying bonds is a common method to earn interest on the money invested. However, there are thousands of bonds in a market, which represent alternatives in multi-criteria decision making problem. Selecting the best bond is not an easy task, and the choice is different depending on the goal of the investor.

Each bond is characterized by several attributes, but we will limit ourself to consideration of only two—the return rate (earning) and the time to maturity (the length of time until the moment when the invested money will be return to the investor along with the interest earned). The return and the maturity time tend to move in the same direction—longer the time to maturity usually leads to the higher expected return—so these two criteria cannot be considered independent. For this reason, the Choquet integral is a good choice for the aggregation operator to combine the values of the return and the time to maturity into a global value of the bond.

In the problem of bond selection, the Shapley values of the return and the time to maturity represent the importance of the each criterion for the investor. In the other words, the investor decides whether the level of the return or the time to maturity should carry bigger weight. For example, if the investor wants to have money available for use in neat future, he/she will give higher Shapley value to the maturity time than to the return rate. The interaction index of degree 2 between two criteria is determined by the correlation between their behaviors. In the case of two attributes considered in this example, the interaction index will have negative value since they are redundant criteria. Of course, it is almost impossible for an investor to give the precise Shapley values and interaction index of degree 2, so the values are given by intervals.

Based on the given Shapley values and interaction indices, the Choquet integral over intervals is calculated for each bond. The resulting interval values are compared by the degree of preference equation (1), and the best bond is selected for the investment.

In general, it is common that higher return of a bond is associated with higher risk of not-getting this return. An investor could exhibit a risk-prone or risk-averse behavior in which case the the first step after calculating Choquet integrals is to determine the interval of importance by equation (4) based on the investor's attitude. Then, the degree of preference equation is applied to the intervals of importance to determine the best bond for the particular investor's behavior.

VI. Conclusion

In this paper, we have presented a rational way of ordering intervals of preference in multicriteria decision making, which is highly needed when evaluating alternatives using the interval-based Choquet integral. In the case when the intervals are disjoint, the ordering of alternatives is a straightforward task. To cope with overlapping intervals, degree of preference was defined to order alternatives.

Moreover, strategies of choice were considered in cases when a decision-maker exhibits risk-prone or risk-averse attitude. A slight modification of the general degree of preference, by calculation of the interval of importance, gives a natural way of ordering intervals of preference that is in agreement with intuitive behavior of the decision-maker.

Finally, a more common situation, where not all parts of the interval are equally probable, was considered. Typically, the interior of the interval has higher chance of giving the correct value than the extreme points, so Gaussian distribution suits the situation much better than generally assumed uniform distribution. A slight modification of the calculation of the interval of preference as well as degree of preference calculation was suggested in order to accommodate this situation.

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