An Interval-valued, 2-additive Choquet Integral for Multi-criteria Decision Making

Martine Ceberio and François Modave
University of Texas at El Paso, CS Dept.
500W University, El Paso, TX, 79968-0518, USA
mceberio@cs.utep.edu, fmodave@cs.utep.edu

Abstract

The aim of this paper is to show how fuzzy measures and intervals can be combined in order to provide a simple and accurate practical solution to multi-criteria decision making problems. More specifically, we construct an interval-based Choquet integral in order to derive preferences over a set of multidimensional alternatives.

Keywords: Decision making, Choquet integral, 2-additivity, interval computation.

1 Introduction

In multi-criteria decision making (MCDM) we aim at ordering multidimensional alternatives. A traditional approach for this problem is to use a simple weighted sum that ensures low complexity and ease of use and where each weight represents the (subjective) importance given by a decision maker to a particular criterion.

Despite its simplicity, this approach suffers a major drawback as we can show that using a weighted sum (or any additive aggregation operator) to evaluate preferences over a set of multidimensional alternatives is equivalent to assuming the independence of criteria (see [6], [8]).

To prevent this problem, non-additive approaches were suggested. More specifically, Fuzzy (or non-additive) measures and integrals can be used to aggregate monodimensional utility functions (see [5]). Until recently, this was done in a rather *ad hoc* way. An axiomatization of multi-criteria decision making using the Choquet integral (a particular case of fuzzy integral) was provided in [7] and [9].

We are now interested in providing a practical solution for such problems using the Choquet integral. That is, under some hypotheses that are not too restrictive, we want to provide an algorithm for ordering multidimensional alternatives. An inherent problem of fuzzy measures is their exponential cost. However, we will show that the notion of 2-additive measures (see [4]) allows us to limit this cost to a $O(n^2)$. Besides, a convenient representation of the Choquet integral w.r.t. to importance and interaction indices (that will be defined further) will allow us to express the Choquet integral in terms of complementary, redundant and independent criteria which is a natural extension of the weighted sum.

In practical problems, we will only require the decision maker to provide importance and interaction indices which are sufficient to define preferences over the alternatives as long as we assume the measure to be 2-additive. This is not very restrictive as generally, it is rather difficult to give a semantic of 3rd and higher order interaction indices. However, it is unlikely that the decision maker can give precise values for these indices. Nevertheless, this is not a major problem as we can reasonably expect the decision maker to be able to give intervals of values.

Therefore, the aim of this paper is to present an interval-valued Choquet integral that will allow us to express intervals of preferences for multidimensional alternatives (under rather weak assumptions). This allows us to have a simple, accurate and implementable model for preferences.

In the first part of this paper, we recall the essentials of multi-criteria decision making and fuzzy measures. Then, we present intervals and their operations, and eventually, we show how to combine these two theories to obtain interval of preferences in the multi-criteria paradigm.

2 Fuzzy measures in MCDM

2.1 Multi-criteria decision making

Let us consider a set $X \subset X_1 \times \cdots \times X_n$. In a multi-criteria decision making problem, the set X represents the set of alternatives, $I = \{1, \cdots, n\}$ is the set of criteria or attributes and the set X_i represents the set of values for attribute i. In general, a decision maker has enough information to order values of attributes in a set X_i. Therefore, we will assume that each set X_i is endowed with a weak order $\succeq_i$. Under a rather weak assumption (namely order-separability), for all $i \in I$, there exists a partial utility function $u_i : X_i \to \mathbb{R}$ such that:

$$\forall x_i, y_i \in X_i \ , \ x_i \succeq_i y_i \Leftrightarrow u_i(x_i) \geq u_i(y_i) \quad (1)$$

In MCDM, we aim at finding a weak order $\succeq$ over X that is "consistent" with the partial orders, that is, we are looking for an aggregation operator $\mathcal{H} : \mathbb{R}^n \to \mathbb{R}$ such that:

$$\forall x, y \in X \ , \ x \succeq y \Leftrightarrow u(x) \succeq u(y)$$

with $x = (x_1, \cdots, x_n) \in X$ and $u(x) = \mathcal{H}(u_1(x_1), \cdots, u_n(x_n))$.

By consistent, we mean that the choice of the aggregation operator should reflect the preferences of the decision maker, and therefore some degree of subjectivity.

A very natural and simple approach for such a problem is to use a simple weighted sum. The decision maker is asked to provide weights $\alpha_i \in [0, 1]$ that reflects the importance of each criterion and such that $\sum_{i=1}^{n} \alpha_i = 1$. The utility function is then defined by:

$$\forall x \in X \ , \ u(x) = \sum_{i=1}^{n} \alpha_i u_i(x_i) \quad (2)$$

Despite an attractive simplicity and low complexity, this approach suffers a major drawback. We can show that using an additive aggregation operator such as a weighted sum is equivalent to assuming all the attributes independent ([8]). In practice, this is not realistic and therefore, we need to turn to non-additive approaches.

2.2 Fuzzy integration

For the sake of our applications, we restrict ourselves to the finite case. However, these definitions can be extended to infinite sets (see [2] and[5] for a detailed presentation of fuzzy integration). A fuzzy integral is a sort of very general averaging operator that can represent the notions of importance of a criterion and interaction between criteria (veto and favor) as we will see thereafter.

To define fuzzy integrals, we need a set of values of importance, this set being the values of the fuzzy measure with respect to which the fuzzy integral is computed. That is, we need a value of importance of each subset of attributes.

In the following definition, $\mathcal{P}(I)$ represents the power set of I.

Definition 1. *Let I be the set of attributes (or any set in a general setting). A set function $\mu : \mathcal{P}(I) \to [0, 1]$ is called a* fuzzy measure *if it satisfies the three following axioms:*

(1) $\mu(\emptyset) = 0$: the empty set has no importance

(2) $\mu(I) = 1$: the maximal set has maximal importance

(3) $\mu(B) \leq \mu(C)$ if $B,\ C \subset I$ and $B \subset C$: a new criterion added cannot make the importance of a coalition (a set of criteria) diminish.

Therefore, in our problem where $card(I) = n$, we need a value for every element of $\mathcal{P}(I)$ that is 2^n values. Considering, the values of the empty set and of the maximal set are fixed, we actually need, $(2^n - 2)$ values or coefficients to define a fuzzy measure. So, there is clearly a trade-off between complexity and accuracy. However, we will see that we can reduce the complexity significantly in order to guarantee that fuzzy measures are used in practical applications.

A fuzzy integral is a sort of weighted mean taking into account the importance of every coalition of criteria.

Definition 2. *Let μ be a fuzzy measure on $(I, \mathcal{P}(I))$ and an application $f : I \rightarrow \mathbb{R}^+$. The* Choquet integral *of f w.r.t μ is defined by:*

$$(C) \int_I f d\mu = \sum_{i=1}^{n} (f(\sigma(i)) - f(\sigma(i-1)))\mu(A_{(i)})$$

where σ is a permutation of the indices in order to have $f(\sigma(1)) \leq \cdots \leq f(\sigma(n))$, $A_{(i)} = \{\sigma(i), \ldots, \sigma(n)\}$ and $f(\sigma(0)) = 0$, by convention.

When there is no risk of confusion, we will write (i) for $\sigma(i)$.

It is easy to see that the Choquet integral is a Lebesgue integral up to a reordering of the indices. Actually, if the fuzzy measure μ is additive, then the Choquet integral reduces to a Lebesgue integral.

2.3 Representation of preferences

We are now able to present how fuzzy measures can be used in lieu of the weighted sum and other more traditional aggregation operators in a multicriteria decision making framework.

It was shown in [9] that under rather general assumptions over the set of alternatives X, and over the weak orders $\succeq_i$, there exists a unique fuzzy measure μ over I such that:

$$\forall x, y \in X \ , \ x \succeq y \Leftrightarrow u(x) \geq u(y) \quad (3)$$

where

$$u(x) = \sum_{i=1}^{n} [u_{(i)}(x_{(i)}) - u_{(i-1)}(x_{(i-1)})]\mu(A_{(i)}) \quad (4)$$

which is simply the aggregation of the monodimensional utility functions using the Choquet integral w.r.t. μ.

Besides, we can show that many aggregation operators can be represented by a Choquet integral (see [5]). This makes the Choquet integral a very general and powerful tool to represent preferences in a multicriteria decision making settings.

However, we are still facing two crucial problems. The proof of the above result is not constructive. That is, it does not provide a fuzzy measure to aggregate monodimensional utilities. Second, as we have said before, evaluating a fuzzy measure requires 2^n values. We are going to see that we can overcome these difficulties and that using fuzzy measures (coupled with intervals) offers a nice solution to multicriteria decision making problems.

Let us start with a couple of definitions that will allow us to show how to limit the complexity to a $O(n^2)$.

The global importance of a criterion is given by evaluating what this criterion brings to every coalition it does not belong to, and averaging this input. This is given by the Shapley value or index of importance (see [11], [3], [4]).

Definition 3. *Let μ be a fuzzy measure over I. The Shapley value of index j is defined by:*

$$v(j) = \sum_{B \subset I \setminus \{j\}} \gamma_I(B)[\mu(B \cup \{j\}) - \mu(B)]$$

with $\gamma_I(B) = \frac{(|I|-|B|-1)! \cdot |B|!}{|I|!}$, $|B|$ denotes the cardinal of B.

The Shapley value can be extended to degree two, in order to define the indices of interactions between attributes (see [4] and [**?**] for the original paper in Japanese).

Definition 4. *Let μ be a fuzzy measure over I. The* interaction index *between i and j is* *defined by:*

$$I(i,j) = \sum_{B\subset I\setminus\{i,j\}} \xi_I(B)\cdot(\mu(B\cup\{i,j\}) - \mu(B\cup\{i\}) - \mu(B\cup\{j\}) + \mu(B))$$

with $\xi_I(B) = \frac{(|I|-|B|-2)!\cdot|B|!}{(|I|-1)!}$.

The interaction indices belong to the interval $[-1,+1]$ and:

- $I(i,j) > 0$ if the attributes i and j are complementary;
- $I(i,j) < 0$ if the attributes i and j are redundant;
- $I(i,j) = 0$ if the attributes i and j are independent.

Interactions of higher orders can also be defined, however we will restrict ourselves to second order interactions which offer a good trade-off between accuracy and complexity. To do so, we define the notion of 2-additive measure.

Definition 5. *A fuzzy measure μ is called* 2-additive *if all its interaction indices of order equal or larger than 3 are null and at least one interaction index of degree two is not null.*

In this particular case of 2-additive measures, we can show that ([4]):

Theorem 1. *Let μ be a 2-additive measure. Then the Choquet integral can be computed by:*

$$(C)\int_I f d\mu = \sum_{I_{ij}>0}(f(i)\wedge f(j))I_{ij} + \sum_{I_{ij}<0}(f(i)\vee f(j))|I_{ij}| + \sum_{i=1}^{n} f(i)(I_i - \tfrac{1}{2}\sum_{j\neq i}|I_{ij}|).$$

Note that this expression justifies the above interpretation of interaction indices, as a positive interaction index corresponds to a conjunction (complementary) and a negative interaction index corresponds to a disjunction (redundant).

In the weighted sum case, we assume that the decision maker can provide us with the weights she/he puts on each criterion. However, we know that this model is inaccurate when trying to deal with dependencies. We could use a Choquet integral instead, as we have seen that they are a convenient and precise tool to model preferences. However, the complexity is very high. Therefore, in order to combine the best of the two worlds, we can ask the decision maker to give the Shapley values, as well as the interaction indices, and then use the reconstruction theorem 1 to obtain the aggregation operator, which is a Choquet integral w.r.t. to a 2-additive measure. Of course, we have to assume the measure to be 2-additive to use theorem 1. However, this is not a serious limitation as the importance and the 2-order interaction are enough to give a thorough semantic interpretation of the results.

Nevertheless, such an approach raises an other problem. How can we expect the decision maker to give a precise value for the importance and interaction indices? In order to overcome this hurdle, we introduce the concept of interval and see how it can be used efficiently to derive "interval of preferences".

3 Intervals

3.1 Real Interval Arithmetic

Interval Arithmetic (IA) is an arithmetic over sets of real numbers called *intervals*. IA has been proposed by Ramon E. Moore [10] in the late sixties in order to model uncertainty, and to tackle rounding errors of numerical computations. For a complete presentation of IA, we refer the reader to [1].

Definition 6 (Real interval). *A* real interval *is a closed and connected set of real numbers. Every real interval $\boldsymbol{x}$ is denoted by $[\underline{x},\overline{x}]$, where its bounds are defined by $\underline{x} = \inf \boldsymbol{x}$ and $\overline{x} = \sup \boldsymbol{x}$. In order to represent the real line with closed sets, $\mathbb{R}$ is compactified in the obvious way with the infinities $\{-\infty,+\infty\}$. The set of real intervals is denoted $\mathbb{I}$.*

Given a subset ρ of $\mathbb{R}$, the *convex hull* of ρ is the real interval $\texttt{Hull}(\rho) = [\inf\rho, \sup\rho]$. The *width* of a real interval $\boldsymbol{x}$ is the real number $w(\boldsymbol{x}) = \overline{x} - \underline{x}$. Given two real intervals $\boldsymbol{x}$ and $\boldsymbol{y}$, $\boldsymbol{x}$ is said to be *tighter than* $\boldsymbol{y}$ if $w(\boldsymbol{x}) \leq w(\boldsymbol{y})$.

Elements of $\mathbb{I}^n$ define boxes. Given $(\boldsymbol{x_1}, \ldots, \boldsymbol{x_n})^T \in \mathbb{I}^n$, the corresponding *box* is the Cartesian product of intervals $\mathbf{X} = \boldsymbol{x_1} \times \cdots \times \boldsymbol{x_n}$. By misuse of notation, the same symbol is used for vectors and boxes. The above-mentioned notions are straightforwardly extended to boxes.

IA operations are set theoretic extensions of the corresponding real operations. Given $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{I}$ and an operation $\diamondsuit \in \{+, -, \times, \div\}$, we have $\boldsymbol{x} \diamondsuit \boldsymbol{y} = \texttt{Hull}\{x \diamondsuit y \mid (x, y) \in \boldsymbol{x} \times \boldsymbol{y}\}$. Due to properties of monotonicity, these operations can be implemented by real computations over the bounds of intervals. For instance, given two intervals $\boldsymbol{x} = [a, b]$ and $\boldsymbol{y} = [c, d]$, we have $\boldsymbol{x} + \boldsymbol{y} = [\ a + c\ ,\ b + d\ ]$.

$$\begin{cases} \boldsymbol{x} + \boldsymbol{y} = [\ a + c\ ,\ b + d\ ] \\ \boldsymbol{x} - \boldsymbol{y} = [\ a - d\ ,\ b - c\ ] \\ \boldsymbol{x} \times \boldsymbol{y} = [\ \min\{ac, ad, bc, bd\}\ , \\ \qquad\qquad \max\{ac, ad, bc, bd\}\ ] \end{cases}$$

The associative law and the commutative law are preserved over $\mathbb{I}$. However, the distributive law does not hold. In general, only a weaker law is verified, called subdistributivity. For all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{I}$, we have:

$$\boldsymbol{x} \times (\boldsymbol{y} + \boldsymbol{z}) \subseteq \boldsymbol{x} \times \boldsymbol{y} + \boldsymbol{x} \times \boldsymbol{z}$$

We observe in particular that equivalent expressions over the real numbers are no longer equivalent when handling intervals: different symbolic expressions may lead to different interval evaluations. This problem is known as the dependency problem of IA, and a fundamental problem in IA consists in finding expressions that lead to tight interval computations.

3.2 Interval Extensions

IA is designed to represent outer approximations of real quantities. The range of a real function f over a domain D, denoted by $\boldsymbol{f}^u(D)$, can be computed by interval extensions.

Definition 7 (Interval extension). *An* interval extension *of a real function $f : D_f \subset \mathbb{R}^n \to \mathbb{R}$ is a function $\boldsymbol{\varphi} : \mathbb{I}^n \to \mathbb{I}$ such that*

$$\forall \mathbf{X} \in \mathbb{I}^n,\ (\mathbf{X} \in D_f \ \Rightarrow \\ \boldsymbol{f}^u(\mathbf{X}) = \{f(\mathrm{x}) \mid \mathrm{x} \in \mathbf{X}\} \subseteq \boldsymbol{\varphi}(\mathbf{X}))\,.$$

This inclusion formula is called Fundamental Theorem of IA*.*

This definition implies the existence of infinitely many interval extensions of a given real function. In particular, the weakest and tightest extensions are respectively defined by: $\mathbf{X} \mapsto [-\infty, +\infty]$ and $\mathbf{X} \mapsto \texttt{Hull}(\boldsymbol{f}^u(\mathbf{X}))$.

The most common extension is known as the *natural extension*. Natural extensions are obtained from the expressions of real functions, and are *inclusion monotonic*[1], which means that given a real function f, its natural extension, denoted $\boldsymbol{f}$, and two intervals $\boldsymbol{x}$ and $\boldsymbol{y}$ such that $\boldsymbol{x} \subset \boldsymbol{y}$, then $\boldsymbol{f}(\boldsymbol{x}) \subset \boldsymbol{f}(\boldsymbol{y})$. Since natural extensions are defined by the syntax of real expressions, two equivalent expressions of a given real function f generally lead to different natural interval extensions. In Figure 1, we see that both interval functions define interval extensions of f. However, one function is clearly better.

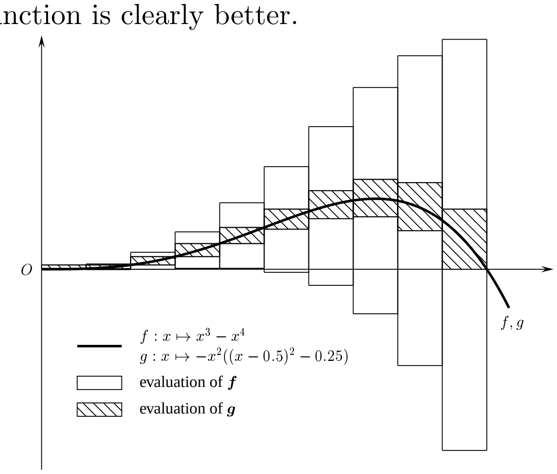

Figure 1: Natural interval evaluations of two expressions of a real function f.

The overestimation problem, known as *dependency problem of IA*, is due to the decorrelation of the occurrences of a variable during interval evaluation. For instance, given $\boldsymbol{x} = [a, b]$ with $a \neq b$, we have $\boldsymbol{x} - \boldsymbol{x} = [a - b, b - a] \supsetneq 0$.

An important result is Moore's theorem known as the *theorem of single occurrences*.

Theorem 2 (Moore [10]). *Let f be a real*

[1] This property follows from the monotonicity of interval operations

function, $f : D_f \subset \mathbb{R}^n \to \mathbb{R}$*, such that* $(x_1, \ldots, x_n) \mapsto t(x_1, \ldots, x_n)$ *where* t *is a symbolic expression interpreted by* f*. If each* x_i *occurs only once in* t*,* $1 \le i \le n$*, then*

$$\forall \mathbf{X} \in \mathbb{I}^n, (\mathbf{X} \subseteq D_f \;\Rightarrow\; \boldsymbol{f}^u(\mathbf{X}) = \boldsymbol{f}(\mathbf{X}))\,.$$

In other words, there is no overestimation if all variables occur only once in the given expression.

Let us remark that interval computations are performed on computers, where real numbers are simulated by floating-point numbers. As a result, real intervals are simulated by real intervals whose bounds are floating-point numbers, called *floating-point intervals*. The set of such intervals is denoted $\mathbb{IF}$. The main difference between $\mathbb{I}$ and $\mathbb{IF}$ is that computations over floating-point numbers need to be rounded.

Floating-Point IA corresponds to Real IA where all intermediate results of interval computations are outward rounded as follows: $[a, b] \in \mathbb{I} \;\;\rightsquigarrow\;\; [\lfloor a \rfloor, \lceil b \rceil] \in \mathbb{IF}$. where $\lfloor a \rfloor$ (resp. $\lceil b \rceil$) is the largest (smallest) element of $\mathbb{F}$ smaller (greater) than or equal to a (b).

For expressions with single occurrences of variables, we commonly say that Moore's theorem is valid except on rounding. The overestimation due to rounding errors has to be distinguished with overestimation that also happens on $\mathbb{I}$.

In the following, the set $\mathbb{IF}$ will be simply denoted by $\mathbb{I}$. Elements of $\mathbb{IF}$ will be called *intervals.*

4 Intervals of preferences

As we have seen before, to define preferences over alternatives, the user is required to provide importance and interaction indices, but is more likely to establish intervals of values than precise values. In this section, we explain how such interval information can be integrated in the scheme of computation of the Choquet integral, by extending its definition to Interval Arithmetic.

Since the user is not longer asked for precise values of indices I_{ij} and I_i, but for intervals[2], we consider intervals of values of these indices, and we respectively denote them $\boldsymbol{I_{ij}}$ and $\boldsymbol{I_i}$, i, $j \in \{1, \ldots, n\}$. As a consequence, the formula for the computation of the Choquet integral is now given by:

$$\begin{aligned}&(C_{\mathbb{I}}) \textstyle\int_I f d\mu = \sum_{\boldsymbol{I_{ij}}>0} (f(i) \wedge f(j)) \boldsymbol{I_{ij}} \\ &+ \textstyle\sum_{\boldsymbol{I_{ij}}<0} (f(i) \vee f(j)) |\boldsymbol{I_{ij}}| + \\ &+ \textstyle\sum_{i=1}^{n} f(i) (\boldsymbol{I_i} - \frac{1}{2} \sum_{j \neq i} |\boldsymbol{I_{ij}}|).\end{aligned}$$

where the annotation $(C_{\mathbb{I}})$ means that the interpretation of this formula is performed using IA. As a consequence, the value of the integral is an interval, which we hope is the tightest one regarding the interval information provided by the user.

However, using IA means that overestimation of the range of real functions may occur, due to the above-mentioned dependency problem of IA. In particular, in the case of Equation 5, every interval variable $\boldsymbol{I_{ij}}$ occurs twice, with different monotonicities (once positively, once negatively), which inevitably leads to overestimating the expected range of values. Therefore, the right part of the formula is rewritten so as to obtain single occurrences only:

$$\begin{aligned}&(C_{\mathbb{I}}) \textstyle\int_I f d\mu = \sum_{i=1}^{n} f(i) \boldsymbol{I_i} \\ &+ \textstyle\sum_{\boldsymbol{I_{ij}}>0} \left((f(i) \wedge f(j)) - \frac{1}{2}(f(i) + f(j))\right) \boldsymbol{I_{ij}} \\ &+ \textstyle\sum_{\boldsymbol{I_{ij}}<0} \left((f(i) \vee f(j)) - \frac{1}{2}(f(i) + f(j))\right) |\boldsymbol{I_{ij}}|\end{aligned} \quad (5)$$

This formula contains only single occurrences of interval variables, which is a guarantee to obtain the exact range of possible values, given the intervals of preferences of the user.

Two alternatives are then compared w.r.t. the corresponding interval values of their interval integral of Choquet, as follows:

$$(C_{\mathbb{I}}) \int_I f d\mu \;\succeq\; (C_{\mathbb{I}}) \int_I g d\mu \;\;\overset{def}{\Leftrightarrow}\;\; \underline{(C_{\mathbb{I}}) \int_I f d\mu} \;\geq\; \overline{(C_{\mathbb{I}}) \int_I g d\mu} \quad (6)$$

This is interpreted as the alternative f is preferred to the alternative g. It is worth noting that if the decision maker gives precise

[2] We will make the assumption (not restrictive) that the decision maker cannot give an interval whose interior contains 0, which would be a contradictory information.

values for the importance and interaction indices, then the interval-based Choquet integral restricts to a standard Choquet integral and the intervals of preferences are real valued numbers.

However, we should also emphasize the fact that the above case is an ideal case where the interval of preferences do not intersect and the preferences are clear. It may happen that:

$$(C_{\mathbb{I}})\int_I f d\mu \bigcap (C_{\mathbb{I}})\int_I g d\mu \neq \emptyset$$

In such a case, we need to define a degree of preference corresponding to the intersection of the intervals.

4.1 Strategy of preference

We could use a trivial solution which is to look at the upper bounds and give preference to the highest upper bound, which corresponds to an optimistic behavior: the preference is given to the alternative more likely to have a high Choquet integral value; or to look at the lower bounds and give preference the the highest lowest bound which then corresponds to a pessimistic behavior: the preference is given to the alternative less likely to have a low Choquet integral value.

However, many alternatives between the very optimistic case and the very pessimistic case are possible. It is our feeling that we need to look simultaneously at the upper and lower bounds as well as the width of the intervals. Indeed, in many situations, the decision maker will exhibit some sort of aversion of risk and will want to have intervals as tight as possible, that is restrict the degree of uncertainty. In particular, we can already draw some strategies of choice as follows.

Suppose that we consider two intervals I and J, corresponding respectively to $(C_{\mathbb{I}})\int_I f d\mu$ and $(C_{\mathbb{I}})\int_I g d\mu$. If the configuration is as illustrated by Figure 2, then an optimistic strategy could consist in giving preference to interval I, since I offers the possibility of having higher Choquet integral values.

It is not as simple when J is included in I. Indeed, when the configuration is as illustrated

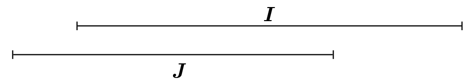

Figure 2: Configuration of Choquet integral intervals I and J where preference is given to interval I.

by Figure 3:

$$\underline{J} - \underline{I} = \overline{I} - \overline{J},$$

without more information, we can guess that there is the same probability for values in I to be smaller than values in J, as to be greater. As a consequence, a reasonable strategy could consist in giving preference to J since J is tighter and therefore more accurate.

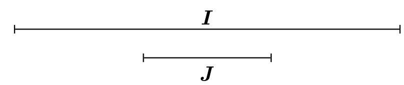

Figure 3: Configuration of Choquet integral intervals I and J where preference is given to interval J.

When interval I is not as well-balanced around J as it was in the previous configuration, two configurations, respectively illustrated by Figures 4 and 5, are to be considered. In such cases, our feeling is that we may have to give preference to the interval that minimizes the risk of having small Choquet integral values. The first case is defined by:

$$\underline{J} - \underline{I} \ > \ \overline{I} - \overline{J}.$$

A safe strategy may consist in preferring J

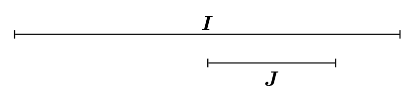

Figure 4: Configuration of Choquet integral intervals I and J where the preference is given to interval J.

for which the probability of obtaining small Choquet integral values is less than for I. On the contrary, the second case is defined by:

$$\underline{J} - \underline{I} \ < \ \overline{I} - \overline{J},$$

and a safe strategy would then consist in preferring I for the same reason as just mentioned.

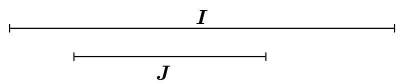

Figure 5: Configuration of Choquet integral intervals I and J where the preference is given to interval I.

As a last remark, probability distribution might be provided by the user for each interval, and attached to the Choquet intervals. As a result, new strategies of choice may be considered. The definition of a degree of preference and a more complete semantics attached to it is part of our future research.

5 Conclusion

In this paper, we have presented a simple computation scheme, combining the Choquet integral (in the 2-additive case) with interval arithmetic that allows us to give intervals of preferences over multidimensional alternatives. The approach is very attractive as it reflects more accurately what we can really expect from a decision maker, yet remains simple and still allows us to represent dependencies between attributes which is not possible with more traditional approaches such as the weighted sum.

In the case where the intervals of preferences are disjoint, the order of alternatives is clearly established. However, it is not as trivial in the (more probable) case where the intervals have an intersection. In this case, some more research is needed to give a consistent ordering of the preferences when the size of the intervals tends to the limit case where there is no uncertainty on the values of importance and interaction.

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