

# Algebraic Product is the Only t-Norm for Which Optimization Under Fuzzy Constraints is Scale-Invariant

Juan Carlos Figueroa Garcia<sup>1</sup>, Martine Ceberio<sup>2</sup>, and Vladik Kreinovich<sup>2</sup>

<sup>1</sup> Universidad Distrital, Departamento de Ingenieria Industrial  
Bogota, Colombia, [filthed@gmail.com](mailto:filthed@gmail.com)

<sup>2</sup> Department of Computer Science, University of Texas at El Paso  
El Paso, TX 79968, USA, [mceberio@utep.edu](mailto:mceberio@utep.edu), [vladik@utep.edu](mailto:vladik@utep.edu)

**Abstract.** In many practical situations, we need to optimize under fuzzy constraints. There is a known Bellman-Zadeh approach for solving such problems, but the resulting solution, in general, depends on the choice of a not well-defined constant  $M$ . We show that this dependence disappears if we use an algebraic t-norm (and-operation)  $f_{\&}(a, b) = a \cdot b$ , and we also prove that the algebraic product is the only t-norm for which the corresponding solution is independent on  $M$ .

## 1 Formulation of the Problem

*Need for optimization under fuzzy constraints.* In decision making, we would like to find the best solution  $x$  among all possible solutions.

For example, if we need to build a chemical plant for producing chemicals needed for space exploration and for sophisticated electronics, then we need to select a design which is the most profitable among all the designs whose possible negative effect on the environment is small. In this example, the objective function is the overall profit.

In this example (and in many similar examples) the objective functions is well defined in the sense that for each alternative  $x$ , we can compute the exact value  $f(x)$  of the objective function for this particular design. In contrast, the constraints are *not* well-defined, they are formulated by using words from a natural language (like “small”), words which are not precise.

A reasonable way to describe the meaning of such imprecise (“fuzzy”) constraints is to use techniques of fuzzy logic (see, e.g., [4, 6, 8]), where to each possible alternative  $x$ , we assign a number  $\mu_c(x)$  describing to what extent this design satisfies the corresponding constraint. To find this value  $\mu_c(x)$ , we can, e.g., ask the user to mark this extent on a scale from 0 to 10, and if the user marks 7, take  $\mu_c(x) = 7/10$ .

This way, the original problem becomes a problem of optimization under fuzzy constraint: find  $x$  for which  $f(x)$  is the largest possible among all  $x$  which satisfy the constraint described by a function  $\mu_c(x)$ .

*Bellman-Zadeh approach to optimization under fuzzy constraints.* To solve such problems, R. Bellman (a known specialist in optimization) and L. Zadeh (the founder of the fuzzy logic approach) came back with the following idea; see, e.g., [1, 4].

First, we (somehow) find the smallest value  $m$  of the objective function  $f(x)$  among all possible solutions  $x$ , and we also find the largest possible value  $M$  of the objective function over all possible constraints. based on the values  $m$  and  $M$ , we can form, for each alternative  $x$ , the degree  $\mu_m(x)$  to which  $x$  is maximal, as  $\mu_m(x) \stackrel{\text{def}}{=} \frac{f(x) - m}{M - m}$ . The larger  $f(x)$ , the larger this degree, and it reaches the value 1 if  $f(x)$  attains the largest possible value  $M$ .

We want to find an alternative which satisfies the constraints *and* optimizes the objective function. In fuzzy techniques, the degree of truth in “and”-statement is approximately described by applying an appropriate *t-norm*  $f(a, b)$  to the degrees to which both statements are true; see, e.g., [4, 6]. A t-norm must satisfy several natural properties: e.g., the fact that  $A \& B$  means the same as  $B \& A$  leads to the commutativity  $f_{\&}(a, b) = f_{\&}(b, a)$ , and the fact that “true”  $\& A$  is equivalent to just  $A$  leads to the property  $f_{\&}(1, a) = a$ .

- By applying the t-norm  $f_{\&}(a, b)$  to the degrees  $\mu_c(x)$  and  $\mu_m(x)$ , we find the degrees  $\mu_s(x) = f_{\&}(\mu_c(x), \mu_m(x))$  to which each alternative  $x$  is a solution.
- We then select the alternative which is the best fit, i.e., for which the degree  $\mu_s(x)$  is the largest.

*Problem: the value  $M$  is not well defined.* Usually, we have some prior experience with similar problems, so we know some alternative(s)  $x$  which were previously selected. The value  $f(x)$  for such “status quo” alternatives can be used as the desired minimum  $m$ .

Finding  $M$  is much more complicated, we do not know which alternatives to include and which not to include. If we replace the original value  $M$  with a new value  $M' > M$ , then the maximizing degree changes, from  $\mu_m(x) = \frac{f(x) - m}{M - m}$  to  $\mu'_m(x) = \frac{f(x) - m}{M' - m}$ . One can easily see that  $\mu'_m(x) = \lambda \cdot \mu_m(x)$  for  $\lambda \stackrel{\text{def}}{=} \frac{M - m}{M' - m} < 1$ .

The problem is that in general, the alternatives for which the functions  $\mu_s(x) = f_{\&}(\mu_c(x), \mu_m(x))$  and  $\mu'_s(x) = f_{\&}(\mu_c(x), \mu'_m(x)) = f_{\&}(\mu_c(x), \lambda \cdot \mu_m(x))$  may be different.

It is therefore desirable to come up with a scheme in which the solution would not change if we simply re-scale  $\mu_m(x)$  by modifying the not well-defined quantity  $M$ .

*What we do in this paper.* In this paper, we show that the dependence on  $M$  disappears if we use algebraic product t-norm  $f_{\&}(a, b) = a \cdot b$ . We also show that this is the only t-norm for which decisions do not depend on  $M$ .

## 2 Main Results

**Definition 1.** By a t-norm, we mean a function  $f_{\&} : [0, 1] \times [0, 1] \rightarrow [0, 1]$  for which  $f_{\&}(a, b) = f_{\&}(b, a)$  and  $f_{\&}(1, a) = a$  for all  $a$  and  $b$ .

*Comment.* Usually, it is also required that the t-norm is associative. However, our results do not need associativity, so they are valid for non-associative and-operations as well; such non-associative operations are sometimes used to more adequately describe human reasoning; see, e.g., [2, 3, 5, 7, 9].

**Definition 2.** Let  $f_{\&}(a, b)$  be a t-norm. We say that optimization under fuzzy constraints is scale-invariant for this t-norm if for every set  $X$ , for every two functions  $\mu_c : X \rightarrow [0, 1]$  and  $\mu_m : X \rightarrow [0, 1]$ , and for every real number  $\lambda \in (0, 1)$ , we have  $S = S'$ , where:

- $S$  is the set of all  $x \in X$  for which the function  $\mu_s(x) = f_{\&}(\mu_c(x), \mu_m(x))$  attains its maximum, i.e., for which  $\mu_s(x) = \max_{y \in X} \mu_s(y)$ ;
- $S'$  is the set of all  $x \in X$  for which the function  $\mu'_s(x) = f_{\&}(\mu_c(x), \lambda \cdot \mu_m(x))$  attains its maximum, i.e., for which  $\mu'_s(x) = \max_{y \in X} \mu'_s(y)$ .

**Proposition 1.** For the algebraic product t-norm  $f_{\&}(a, b) = a \cdot b$ , optimization under fuzzy constraints is scale-invariant.

**Proposition 2.** The algebraic product t-norm  $f_{\&}(a, b) = a \cdot b$  is the only t-norm for which optimization under fuzzy constraints is scale-invariant.

*Proof of Proposition 1.* For the algebraic product t-norm:

- $S$  is the set of all  $x \in X$  for which the function  $\mu_s(x) = \mu_c(x) \cdot \mu_m(x)$  attains its maximum, and
- $S'$  is the set of all  $x \in X$  for which the function  $\mu'_s(x) = \mu_c(x) \cdot \lambda \cdot \mu_m(x)$  attains its maximum.

Here,  $\mu'_s(x) = \lambda \cdot \mu_s(x)$  for a positive number  $\lambda$ . Clearly,  $\mu_s(x) \geq \mu_s(y)$  if and only if  $\lambda \cdot \mu_s(x) \geq \lambda \cdot \mu_s(y)$ , so the optimizing sets  $S$  and  $S'$  indeed coincide.

*Proof of Proposition 2.* Let  $f_{\&}(a, b)$  be a t-norm for which optimization under fuzzy constraints is scale-invariant, and let  $a$  and  $b$  be two number from the interval  $[0, 1]$ . Let us prove that  $f_{\&}(a, b) = a \cdot b$ .

Let us consider  $X = \{x_1, x_2\}$  with  $\mu_c(x_1) = \mu_m(x_2) = a$  and  $\mu_c(x_2) = \mu_m(x_1) = 1$ . In this case,  $\mu_s(x_1) = f_{\&}(\mu_c(x_1), \mu_m(x_1)) = f_{\&}(a, 1)$ . Due to commutativity, we get  $\mu_s(x_1) = f_{\&}(1, a)$  and due to the second property of the t-norm, we get  $\mu_s(x_1) = a$ .

Similarly, we have  $\mu_s(x_2) = f_{\&}(\mu_c(x_2), \mu_m(x_2)) = f_{\&}(1, a)$ . Due to the second property of the t-norm, we also get  $\mu_s(x_2) = a$ .

Since  $\mu_s(x_1) = \mu_s(x_2)$ , the optimizing set  $S$  consists of both elements  $x_1$  and  $x_2$ .

Due to scale-invariance, for  $\lambda = b$ , the same set  $S' = S = \{x_1, x_2\}$  must be the optimizing set for the function  $\mu'_s(x) = f_{\&}(\mu_c(x), \lambda \cdot \mu_m(x))$ . Thus, we must have  $\mu'_s(x_1) = \mu'_s(x_2)$ , i.e.,  $f_{\&}(a, b \cdot 1) = f_{\&}(1, b \cdot a)$ . So,  $f_{\&}(a, b) = f_{\&}(1, a \cdot b)$ . Due to the second property of the t-norm, we conclude that  $f_{\&}(a, b) = a \cdot b$ .

The proposition is proven.

*Acknowledgments.* This work was supported in part by the National Science Foundation grants 0953339, HRD-0734825 and HRD-1242122 (Cyber-ShARE Center of Excellence) and DUE-0926721, by Grants 1 T36 GM078000-01 and 1R43TR000173-01 from the National Institutes of Health, and by a grant N62909-12-1-7039 from the Office of Naval Research.

This work was performed when Juan Carlos Figueroa Garcia was a visiting researcher at the University of Texas at El Paso.

## References

1. R. E. Bellman and L. A. Zadeh, "Decision making in a fuzzy environment", *Management Science*, 1970, Vol. 47, No. 4, pp. B141–B145.
2. B. Bouchon-Meunier, V. Kreinovich, and H. T. Nguyen, "Non-Associative Operations", *Proceedings of the Second International Conference on Intelligent Technologies InTech'2001*, Bangkok, Thailand, November 27–29, 2001, pp. 39–46.
3. I. R. Goodman, R. A. Trejo, V. Kreinovich, J. Martinez, and R. Gonzalez, "An even more realistic (non-associative) interval logic and its relation to psychology of human reasoning", *Proceedings of the Joint 9th World Congress of the International Fuzzy Systems Association and 20th International Conference of the North American Fuzzy Information Processing Society IFSA/NAFIPS'2001*, Vancouver, Canada, July 25–28, 2001, pp. 1586–1591.
4. G. J. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic*, Prentice Hall, Upper Saddle River, New Jersey, 1995.
5. J. Martinez, L. Macias, A. Esper, J. Chaparro, V. Alvarado, S. A. Starks, and V. Kreinovich, "Towards more realistic (e.g., non-associative) and- and or-operations in fuzzy logic", *Proceedings of the 2001 IEEE Systems, Man, and Cybernetics Conference*, Tucson, Arizona, October 7–10, 2001, pp. 2187–2192.
6. H. T. Nguyen and E. A. Walker, *First Course In Fuzzy Logic*, CRC Press, Boca Raton, Florida, 2006.
7. R. Trejo, V. Kreinovich, I. R. Goodman, J. Martinez, and R. Gonzalez, "A Realistic (Non-Associative) Logic And a Possible Explanations of 7±2 Law", *International Journal of Approximate Reasoning*, 2002, Vol. 29, pp. 235–266.
8. L. A. Zadeh, "Fuzzy sets", *Information and Control*, 1965, Vol. 8, pp. 338–353.
9. H. H. Zimmerman and P. Zysno, "Latent connectives in human decision making", *Fuzzy Sets and Systems*, 1980, Vol. 4, pp. 37–51.