Continuous If-Then Statements Are Computable

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Abstract. In many practical situations, we must compute the value of an if-then expression $f(x)$ defined as "if $c(x) \geq 0$ then $f_+(x)$ else $f_-(x)$", where $f_+(x)$, $f_-(x)$, and $c(x)$ are computable functions. The value $f(x)$ cannot be computed directly, since in general, it is not possible to check whether a given real number $c(x)$ is non-negative or non-positive. Similarly, it is not possible to compute the value $f(x)$ if the if-then function is discontinuous, i.e., when $f_+(x_0) \neq f_-(x_0)$ for some x_0 for which $c(x_0) = 0$.

In this paper, we show that if the if-then expression is continuous, then we can effectively compute $f(x)$.

Practical need for if-then statements. In many practical situations, we have different models for describing a phenomenon:

– a model $f_+(x)$ corresponding to the case when a certain constraint $c(x) \geq 0$ is satisfied, and
– a model $f_-(x)$ corresponding to the case when this constraint is not satisfied, i.e., when $c(x) < 0$ (usually, the second model is also applicable when $c(x) \leq 0$).

For example, in Newton's gravitation theory, when we are interested in the gravitation force generated by a celestial body – i.e., approximately, a sphere of a certain radius R – we end up with two different formulas:

– a formula $f_+(x)$ that describes the force outside the sphere, i.e., where

$$c(x) \stackrel{\text{def}}{=} \|\boldsymbol{r}\| - R \geq 0,$$

and
– a different formula $f_-(x)$ that describes the force inside the sphere, i.e., where

$$c(x) = \|\boldsymbol{r}\| - R \leq 0.$$

Towards a precise formulation of the computational problem. In such situations, we have the following problem:

- we know how to compute the functions $f_+(x)$, $f_-(x)$, and $c(x)$;
- we want to be able to compute the corresponding "if-then" function

$$f(x) \stackrel{\text{def}}{=} \text{ if } c(x) \geq 0 \text{ then } f_+(x) \text{ else } f_-(x).$$

In general, we say that a function $f(x)$ is *computable* if there is an algorithm that, given an input x and a rational number $\varepsilon > 0$, produces a rational number r for which $|f(x) - r| \leq \varepsilon$.

In the above formulation, we assume that the function $c(x)$ is computable for all possible values x from a given set X, and that:

- the function $f_+(x)$ is computable for all values $x \in X$ for which $c(x) \geq 0$; and
- the function $f_-(x)$ is computable for all values $x \in X$ for which $c(x) \leq 0$.

Why this problem is non-trivial. The value $f(x)$ cannot be computed directly, since in general, it is not possible to check whether a given real number $c(x)$ is non-negative or non-positive; see, e.g., [2, 3].

Discontinuous if-then statements are not computable. It is known that every computable function is everywhere continuous; see, e.g., [3].

Thus, when the if-then function $f(x)$ is not continuous, i.e., when $f_+(x_0) \neq f_-(x_0)$ for some x_0 for which $c(x_0) = 0$, then the function $f(x)$ is not computable.

Our main result. In this paper, we show that in all other cases, i.e., when the if-then function $f(x)$ is continuous, it is computable.

Algorithm: main idea. The main idea behind our algorithm is that in reality, we have one of the three possible cases:

- case of $c(x) > 0$, when $f(x) = f_+(x)$;
- case of $c(x) < 0$, when $f(x) = f_-(x)$; and
- case of $c(x) = 0$, when $f(x) = f_+(x) = f_-(x)$.

Let us analyze these three cases one by one.

In the first case, let us compute $c(x)$ with higher and higher accuracy $\varepsilon = 2^{-k}$, $k = 1, 2, \ldots$ As soon as we reach the accuracy $2^{-k} < \dfrac{c(x)}{2}$, for which $c(x) > 2 \cdot 2^{-k}$, we get an approximation r_k for which $|c(x) - r_k| \leq 2^{-k}$, i.e., for which

$$r_k > c(x) - 2^{-k} \geq 2 \cdot 2^{-k} - 2^{-k} = 2^{-k}$$

and thus, $r_k > 2^{-k}$. Since we know that $c(x) \geq r_k - 2^{-k}$, we thus conclude that $c(x) > 0$.

Similarly, in the second case, if we compute $c(x)$ with higher and higher accuracy 2^{-k}, we will reach an accuracy $2^{-k} < \frac{|c(x)|}{2}$, for which the corresponding approximate value r_k satisfy the inequality $r_k < -2^{-k}$ and thus, we can conclude that $c(x) < 0$.

In the third case, since $f_+(x) = f_-(x)$, if we compute $f_+(x)$ and $f_-(x)$ with accuracy $\varepsilon > 0$, then the resulting approximate values r_+ and r_- satisfy the inequalities $|f(x) - r_+| = |f_+(x) - r_+| \le \varepsilon$ and $|f(x) - r_-| = |f_-(x) - r_-| \le \varepsilon$ and therefore,

$$|r_+ - r_-| \le |r_+ - f(x)| + |f(x) - r_-| \le \varepsilon + \varepsilon = 2\varepsilon.$$

Vice versa, if the inequality $|r_+ - r_-| \le 2\varepsilon$ is satisfied (even if we know nothing about $c(x)$), then in reality, the value $f(x)$ coincides wither with $f_+(x)$ or with $f_-(x)$.

In the first subcase, when $f(x) = f_+(x)$, we have

$$|f(x) - r_+| = |f_+(x) - r_+| \le \varepsilon$$

and

$$|f(x) - r_-| = |f_+(x) - r_-| \le |f_+(x) - r_+| + |r_+ - r_-| \le \varepsilon + 2\varepsilon = 3\varepsilon.$$

Thus, due to convexity of the absolute value, we have

$$|f(x) - \overline{r}| \le \frac{1}{2} \cdot (|f(x) - r_+| + |f(x) - r_-|) \le \frac{\varepsilon + 3\varepsilon}{2} = 2\varepsilon.$$

In the second subcase, when $f(x) = f_-(x)$, we have

$$|f(x) - r_-| = |f_-(x) - r_-| \le \varepsilon$$

and

$$|f(x) - r_+| = |f_-(x) - r_+| \le |f_-(x) - r_-| + |r_- - r_+| \le \varepsilon + 2\varepsilon = 3\varepsilon.$$

Thus, due to convexity of the absolute value, we have

$$|f(x) - \overline{r}| \le \frac{1}{2} \cdot (|f(x) - r_-| + |f(x) - r_+|) \le \frac{\varepsilon + 3\varepsilon}{2} = 2\varepsilon.$$

In both case, we have $|f(x) - \overline{r}| \le 2\varepsilon$. So, if we want to compute $f(x)$ with a given accuracy $\alpha > 0$, it is sufficient to find $\frac{\alpha}{2}$-approximations r_- and r_+ to $f_-(x)$ and $f_+(x)$ for which $|r_+ - r_-| \le \alpha$

Thus, we arrive at the following algorithm for computing the if-then function $f(x)$.

Resulting algorithm. To compute $f(x)$ with a given accuracy α, we simultaneously run the following three processes:

– computing $c(x)$ with higher and higher accuracy $\varepsilon = 2^{-k}$, $k = 1, 2, \ldots$;
– computing $f_-(x)$ with accuracy $\frac{\alpha}{2}$; and
– computing $f_+(x)$ with accuracy $\frac{\alpha}{2}$.

Let us denote:

– the result of computing $c(x)$ with accuracy 2^{-k} by r,
– the result of the second process by r_-, and
– the result of the third process by r_+.

As we have mentioned in our analysis, eventually, one of the following three events will happen:

– either we find out that $r_k > 2^{-k}$; in this case we know that ($c(x) > 0$ and hence) the third process will finish, so we finish it and return r_+ as the desired α-approximation to $f(x)$;
– or we find out that $r_k < -2^{-k}$; in this case we know that ($c(x) < 0$ and hence) the second process will finish, so we finish it and return r_- as the desired α-approximation to $f(x)$;
– or we find out that $|r_+ - r_i| \le \alpha$; in this case, we return $\overline{r} = \frac{r_- + r_+}{2}$ as the desired α-approximation to $f(x)$.

Historical comment. Our proof is a simplified version of the proofs described, in a more general setting, in [3]; see also [1].

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