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On the Normalization of Interval-Based Possibility Distributions

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Abstract

Possibilistic logic is an important framework for representing and reasoning with uncertain and inconsistent pieces of information. Standard possibilistic logic expressions are propositional logic formulas associated with positive real degrees belonging to [0,1]. Recently, a flexible representation of uncertain information, where the weights associated with formulas or possible worlds are in the form of intervals, has been proposed. This paper focuses on the problem of normalization of interval-based possibility distributions. We provide a natural procedure to normalize a sub-normalized interval-based possibility distribution. This procedure is based on the concept of normalized compatible and standard possibility distributions.

Introduction

Possibilistic logic (e.g, (Lang 2001; Dubois and Prade 2004)) is a well-known framework for dealing with uncertainty and reasoning under inconsistent knowledge bases. Uncertainty is syntactically represented by a set of weighted formulas of the form K={(φ_i, α_i): i=1, .., n} where φ_i's are propositional formulas and α_i's are real numbers belonging to [0,1]. The pair (φ_i, α_i) means that φ_i is certain (or important) to at least a degree α_i. Uncertainty is also represented at the semantic level by associating a possibility degree with each possible world (or interpretation). A standard possibility distribution is said to be normalized if there exists at least one interpretation which is fully consistent, namely having a possibility degree of 1 (Dubois 2006; 2014).

Interval-based uncertainty representations (e.g, interval-based probabilities) are well-known frameworks for encoding, reasoning and decision making with poor information, ill-known and imprecise beliefs, confidence intervals, multi-source information, etc. (Nguyen and Kreinovich 2014; Dubois 2006). The framework considered in this paper is the one of interval-based possibilistic logic (Benferhat et al. 2011). At the syntactic level, pieces of information are represented by an interval-based possibilistic knowledge base, of the form $IK = \{(\varphi_i, I_i) : i = 1, .., n\}$ where I_i is a closed sub-interval of $[0, 1]$. The pair (φ_i, I_i), called an interval-based weighted formula, means that the weight associated with φ_i is one of the elements in I_i.

Similarly, the semantic of interval-based possibilistic logic is an interval-based possibility distribution, denoted by $I\pi$, where a sub-interval of [0,1] is assigned to each interpretation or each possible world. An interval-based possibility distribution is viewed as a family of standard compatible possibility distributions obtained by combining all possible values of intervals.

This paper focuses on the semantics part of possibilistic logic and addresses the problem of normalizing a sub-normalized interval-based possibility distribution. The normalization problem appears for instance when merging several standard or interval-based possibility distributions issued from different sources. Even if it is reasonable to require that each individual possibility distribution is normalized, it is unlikely that their fusion leads to a normalized distribution. We propose a natural procedure which consists in applying standard (min-based or product-based) normalization on the set of all compatible standard possibility distributions associated with an interval-based possibility distribution. We show that for the min-based normalization, the obtained result is no longer an interval-based possibility distribution. However, when the product-based normalization is used, then the result is indeed a normalized interval-based possibility distribution. We provide the computation of lower and upper endpoints associated with the result of normalizing a sub-normalized interval-based possibility distribution.

The rest of this paper is organized as follows: Section 2 presents a brief refresher on possibilistic logic and interval-based possibilistic logic. Section 3 presents the two forms of normalization that may exist for interval-based possibility distributions and discusses the case of degenerate possibility distributions. Section 4 analyzes the compatible-based normalization procedures of an interval-based possibility distribution. Section 5 contains the main conclusions.

2. A brief refresher on standard and interval-based possibilistic logic

Possibility theory is an alternative uncertainty theory suited for representing and reasoning with uncertain and incomplete information (Dubois 2006; 2014). The concept of possibility distribution π is a fundamental building block of possibility theory; it is a function from the set of possible worlds or interpretations Ω to $[0, 1]$. $\pi(\omega)$ represents the degree of consistency (or feasibility) of the interpretation ω with respect to the available knowledge. $\pi(\omega)$=1 means that ω is fully consistent with the available knowledge, while $\pi(\omega)$=0 means that ω is impossible. $\pi(\omega)>\pi(\omega')$ simply means that ω is more consistent or more feasible than ω'. Note that possibility degrees are interpreted either i) *qualitatively* (in min-based possibility theory) where only the "ordering" of the values is important, or *quantitatively* (in product-based possibility theory where the possibilistic scale [0,1] is numerical and one of the possible interpretations of quantitative possibility degrees is viewing $\pi(\omega)$ as degrees of surprise as in ordinal conditional functions (Spohn 1988).

Another important concept in possibility theory is the one of possibility measure, denoted $\Pi(\varphi)$, and computing the possibility degree of an event $\varphi \subseteq \Omega$. It is defined as follows:

$$\Pi(\varphi) = \max_{\omega \in \varphi}(\pi(\omega)).$$

The necessity measure is the dual of possibility measure and evaluates the certainty implied by the current knowledge of the world. Namely, $N(\varphi)$=$1-\Pi(\overline{\varphi})$ where $\overline{\varphi}$ denotes the complement of φ.

A possibility distribution π is said to be normalized if there exists an interpretation ω such that $\pi(\omega)$=1; it is said to be sub-normalized otherwise. Sub-normalized possibility distributions encode inconsistent sets of beliefs or constraints.

Standard possibilistic logic

We consider a finite propositional language $\mathcal{L}$. We denote by Ω the finite set of interpretations of $\mathcal{L}$ (universe of discourse), and by ω an element of Ω. A possibilistic formula is a pair (φ_i, α_i) where φ is an element of $\mathcal{L}$ and $\alpha \in [0, 1]$ is a valuation of φ representing $N(\varphi)$. A possibilistic base K=$\{(\varphi_i, \alpha_i) : i = 1, .., n\}$ is then a set of possibilistic formulas. Possibilistic knowledge bases are well-known compact representations of possibility distributions. Each piece of information (φ_i, α_i) from a possibilistic knowledge base can be viewed as a constraint which restricts a set of possible interpretations. If an interpretation ω satisfies φ_i then its possibility degree is equal to 1 (ω is completely compatible with the belief φ_i), otherwise it is equal to $1-\alpha_i$ (the more φ_i is certain, the less ω is possible). In particular, if α_i=1, then the degree of any interpretation falsifying φ_i is equal to 0, namely is impossible. More generally, given a possibilistic base K, we can generate a unique possibility distribution where interpretations ω satisfying all the propositional formulas in K have the highest possible degree $\pi(\omega)$=1 (since they are fully consistent), whereas the others are pre-ordered with respect to the highest formulas they falsify. More formally: $\forall \omega \in \Omega$,

$$\pi_K(\omega) = \begin{cases} 1 \text{ if } \forall (\varphi_i, \alpha_i) \in K, \omega \models \varphi; \\ 1 - \max\{\alpha_i : (\varphi_i, \alpha_i) \in K, \omega \nvDash \varphi_i\} & \text{otherwise.} \end{cases} \quad (1)$$

Interval-based possibilitic logic

This subsection gives a brief refresher on interval-based possibilistic logic (Benferhat et al. 2011) where uncertainty is not described with single values but by intervals of possible degrees. We use closed sub-intervals $I \subseteq [0, 1]$ to encode the uncertainty associated with formulas or interpretations. If I is an interval, then we denote by $\lceil I \rceil$ and $\lfloor I \rfloor$ its upper and lower endpoints respectively.

Compatible possibility distributions An interval-based possibility distribution, denoted by $I\pi$, is a function from Ω to $\mathcal{I}$. $I\pi(\omega)$=I means that the possibility degree of ω is one of the elements of I. An interval-based possibility distribution is viewed as a family of compatible standard possibility distributions defined by:

Definition 1 (Compatible possibility distributions). *Let $I\pi$ be an interval based possibility distribution. A possibility distribution π is said to be compatible with $I\pi$ iff $\forall \omega \in \Omega$, $\pi(\omega) \in I\pi(\omega)$.*

Of course, compatible distributions are not unique. We denote by $\mathcal{C}(\mathcal{I}\pi)$ the set of all compatible possibility distributions with $I\pi$. Example 1 gives an example of interval-based possibility distribution $I\pi$. Standard possibility distributions π_1 and π_2 are compatible while possibility distribution π_3 is not compatible because $\pi_3(\omega_3) \notin I\pi(\omega_3)$.

Example 1. *Let $I\pi$ be an interval-based possibility distribution over Ω=$\{\omega_1, \omega_2, \omega_3\}$.*

ω	$I\pi(\omega)$
ω_1	[.7, 1]
ω_2	[.6, .8]
ω_3	[.4, .5]

ω	$\pi_1(\omega)$	ω	$\pi_2(\omega)$	ω	$\pi_3(\omega)$
ω_1	1	ω_1	1	ω_1	1
ω_2	.7	ω_2	.6	ω_2	.7
ω_3	.4	ω_3	.5	ω_3	**1**

Table 1: Example of an interval-based possibility distribution $I\pi$, two compatible distributions π_1 and π_2 and a non compatible distribution π_3.

Definition 1 gives minimal requirements for the notion of compatible possibility distributions. One may additionally require that a compatible possibility distribution should be normalized as it is done in interval-based probability distributions (Walley 1991). Since the problem considered in the paper is the one of normalizing an interval-based possibility distribution, such requirement is not added. Of course, if there exists an interpretation ω such that $I\pi(\omega)$=$[1, 1]$ then such additional requirement is always satisfied as in *Example 1*.

The syntactic representation of interval-based possibilistic logic generalizes the notion of a possibilistic base to an interval-based possibilistic knowledge base.

Definition 2 (Interval-based possibilistic base). *An interval-based possibilistic base, denoted by IK, is a multiset of formulas associated with intervals:*

$$IK = \{(\varphi,\ I), \varphi \in \mathcal{L} \text{ and } I \text{ is a closed sub-interval of } [0,1]\}$$

The intuitive interpretation of (φ, I) is that the certainty degree of φ is one of the elements of I.

From interval-based possibilistic bases to interval-based possibility distributions

As in standard possibilistic logic, an interval-based knowledge base IK is also a compact representation of an interval-based possibility distribution π_{IK}.

Definition 3 (Interval-based possibility distribution). *Let IK be an interval-based possibilistic base, then:*

$$\pi_{IK}(\omega) = [\lfloor \pi_{IK}(\omega) \rfloor, \lceil \pi_{IK}(\omega) \rceil]$$

where:

$$\lfloor \pi_{IK}(\omega) \rfloor = \begin{cases} 1 \text{ if } \forall (\varphi, I) \in IK,\ \omega \models \varphi \\ 1 - max\{\lceil I \rceil : (\varphi_i, I) \in K, \omega \nvDash \varphi_i\} \text{ otherwise.} \end{cases}$$

and

$$\lceil \pi_{IK}(\omega) \rceil = \begin{cases} 1 \text{ if } \forall (\varphi, I) \in IK,\ \omega \models \varphi \\ 1 - max\{\lfloor I \rfloor : (\varphi_i, I) \in K, \omega \nvDash \varphi_i\} \text{ otherwise.} \end{cases}$$

Definition 3 clearly extends the one given by Equation 1 when $\lfloor I \rfloor = \lceil I \rceil$, namely when intervals associated with formulas are singletons.

Remark: In (Benferhat et al. 2011) it is assumed that the lower endpoint $\lfloor I \rfloor$ associated with propositional formulas should be strictly positive. In this paper, intervals with nul lower endpoints are accepted. An interval of the form $[0, \lceil I \rceil]$, associated with a formula φ, means that either φ is not believed (the source that provided the formula is not reliable) or it is believed to at most a degree $\lceil I \rceil$.

Example 2. *Let $IK = \{(a, [.5, .7]), (a \vee b, [.6, .9]), (a \wedge c, [.2, .4])\}$ be an interval-based possibilistic base where a, b and c are three symbols of a propositional language $\mathcal{L}$. The interval-based possibility distribution corresponding to IK according to Definition 3 is given in Table 2.*

ω	$I\pi(\omega)$
abc	[1, 1]
$ab\neg c$	[.6, .8]
$a\neg bc$	[1, 1]
$a\neg b\neg c$	[.6, .8]
$\neg abc$	[.3, .5]
$\neg ab\neg c$	[.3, .5]
$\neg a\neg bc$	[.1, .4]
$\neg a\neg b\neg c$	[.1, .4]

Table 2: Example of interval-based possibility distribution induced by an interval-based possibilistic base.

Semantic normalization

Weak and strong normalized interval-based possibility distributions

In standard possibility theory, a possibility distribution π is said to be normalized if there exists an interpretation ω such that $\pi(\omega)=1$. This reflects the presence of an interpretation (or a solution) that is fully coherent (or compatible, satisfactory) with respect to the set or available knowledge (or constraints, preferences). A possibility distribution π is said to be sub-normalized if:

$$h(\pi) = \max\{\pi(\omega) : \omega \in \Omega\} \quad (2)$$

is less than 1. $h(\pi)$ is called the normalization degree of π.

If π is sub-normalized then there are two main ways to normalize a sub-normalized possibility distribution π:

- **(product-based normalization)** either we shift up proportionally all the interpretations, and we get:

$$\circ_P(\pi)(\omega) = \frac{\pi(\omega)}{h(\pi)} \quad (3)$$

- **(min-based normalization)** or we only increase the degrees of the best interpretations until reaching the degree 1, and we get:

$$\circ_B(\pi)(\omega) = \begin{cases} 1 & \text{if } \pi(\omega) = h(\pi) \\ \pi(\omega) & \text{otherwise} \end{cases} \quad (4)$$

When we deal with the interval-based possibility theory, there are two natural ways to define a normalized interval-based possibility distribution.

Weak normalization: An interval-based possibility distribution $I\pi$ is said to be weakly normalized if there exists an interpretation ω such that $\lceil I\pi(\omega) \rceil = 1$. In terms of compatible possibility distributions, a weak normalization guarantees the existence of at least one normalized possibility distribution π which is compatible with an interval-based distribution.

Strong normalization: An interval-based possibility distribution $I\pi$ is said to be strongly normalized if there exists an interpretation ω such that $\pi(\omega)=[1, 1]$. In terms of compatible possibility distributions, a strong normalization requires that all the compatible possibility distributions, of an interval-based possibility distribution, should be normalized.

Sub-normalized interval-based possibility distributions are those that are not weakly normalized. More precisely,

Definition 4 (Subnormalized distribution). *An interval-based possibility distribution is said to be sub-normalized if there is no interpretation ω such that $\lceil I\pi(\omega) \rceil = 1$.*

In the following, we use again, for the sake of simplicity, $h(I\pi)$ to denote the normalization interval of $I\pi$ defined by:

$$h(I\pi) = \Big[max\{\lfloor I\pi(\omega) \rfloor : \omega \in \Omega\}, max\{\lceil I\pi(\omega) \rceil : \omega \in \Omega\}\Big] \quad (5)$$

Degenerate interval-based possibility distribution

Before presenting how to normalize a sub-normalized interval-based possibility distribution, let us consider the degenerate case where $\forall \omega \in \Omega, I\pi(\omega) = [0, 0]$. Such interval-based possibility distribution expresses a very strong conflict. A normalization of such degenerate interval-based possibility distribution gives:

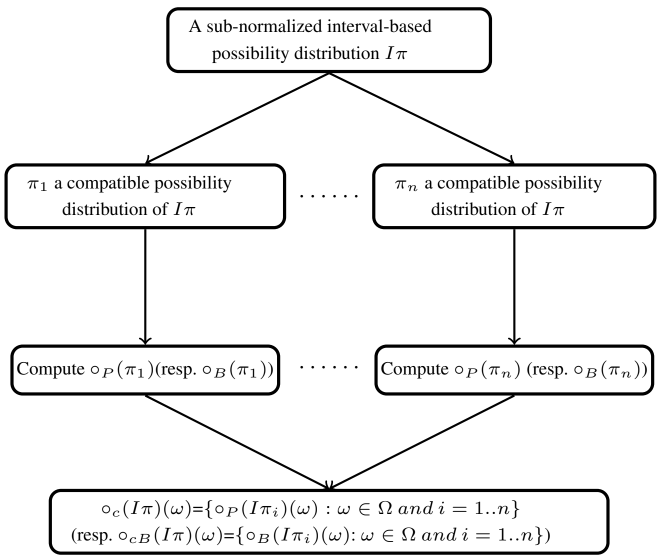
$$\forall \omega \in \Omega, \circ_c(I\pi)(\omega) = [1, 1].$$

$\circ_c(I\pi)$ reflects total ignorance situations.

In the rest of this paper, it is assumed that all interval-based possibility distributions $I\pi$ are not degenerate, namely there exists at least one interpretation such that $\lceil I \rceil > 0$.

Compatible-based normalization

A natural way to define $\circ_c(I\pi)$ is to simply use the concept of compatible possibility distributions, as it is illustrated by the following figure.



To be more precise, let $I\pi$ be a sub-normalized possibility distribution and $\mathcal{C}(\mathcal{I}\pi)$ be the set of compatible possibility distributions of $I\pi$. Then, for best-based normalization define :

$$\circ_{cB}(I\pi)(\omega) = \{\circ_B(\pi)(\omega) : \omega \in \Omega, \pi \in \mathcal{C}(\mathcal{I}\pi)\}, \qquad (6)$$

where $\circ_B(\pi)$ is given by Equation 4.

Equation 6 means that in order to normalize an interval-based possibility distribution $I\pi$: i) first generate all the compatible possibility distributions, ii) then normalize each of these compatible distributions using Equation 4, and iii) lastly collect all the possible degrees of an interpretation ω to obtain $\circ_{cB}(I\pi)(\omega)$.

Unfortunately, when using min-based normalization $\circ_{cB}(I\pi)(\omega)$ is not an interval as it is shown by the following counter-example:

Example 3. *The following table contains an example of a sub-normalized interval-based possibility distribution :*

$\omega_i \in \Omega$	$I\pi(\omega)$
ω_1	$[.8, .9]$
ω_2	$[.5, .8]$
ω_3	$[0, 0]$
ω_4	$[0, 0]$

One can check that for every compatible distribution π such that $\pi(\omega_2) \in [.5, .8[$ we have $\circ_B(\pi)(\omega_2) \in [.5, .8[$ (since $\pi(\omega_1) \geq .8$). Now for compatible possibility distributions where $\pi(\omega_2) = .8$ we have either $\circ_B(\pi)(\omega_2) = .8$ (if $\pi(\omega_1) > .8$) or $\circ_B(\pi)(\omega_2) = 1$ (if $\pi(\omega_1) = .8$). Hence : $\circ_B(\pi)(\omega_2) = [.5, .8] \cup \{1\}$.

The situation is different when using propositional-based normalization (Equation 3) instead of best-based normalization (Equation 4). Namely, define:

$$\circ_c(I\pi)(\omega) = \{\circ_P(\pi)(\omega) : \omega \in \Omega, \pi \in \mathcal{C}(\mathcal{I}\pi)\} \qquad (7)$$

In this case, for each $\omega \in \Omega$, $\circ_c(I\pi)(\omega)$ is an interval as it is shown by the following proposition.

Proposition 1. *Let $I\pi$ be a sub-normalized interval-based possibility distribution. Then $\circ_c(I\pi)$ given by Equation 7 is an interval-based possibility distribution.*

Proof. Let us show that $\circ_c(I\pi)(\omega)$ is indeed an interval. Assume that there exist two numbers α and β such that:

- $\alpha < \beta$,
- $\alpha \in \circ_c(I\pi)(\omega)$, $\beta \in \circ_c(I\pi)(\omega)$, and
- $\forall \gamma$ such that $\alpha < \gamma < \beta$ we have $\gamma \notin \circ_c(I\pi)(\omega)$.

The assumption $\alpha \in \circ_c(I\pi)(\omega)$ means that there exists a compatible possibility distribution π such that :

$$\alpha = \frac{\pi(\omega)}{h(\pi)} .$$

Since $\alpha < \beta$ then trivially there exists ε such that:

$$\alpha = \frac{\pi(\omega)}{h(\pi)} < \frac{\pi(\omega) + \varepsilon}{h(\pi)} < \beta.$$

Therefore it is enough to define a new compatible possibility distribution π' such that $\pi'(\omega)=\pi(\omega)+\varepsilon$ and $\forall \omega'$, $\pi'(\omega')=\pi(\omega')$. Clearly, π' is compatible and $\circ_P(\pi'(\omega))=\dfrac{\pi(\omega) + \varepsilon}{h(\pi)} \in \circ_c(I\pi)(\omega)$.

□

It remains to specify the lower and upper endpoints of $\circ_c(I\pi)(\omega)$. Recall that, from Equation 3, for a given sub-normalized compatible possibility distribution π we have $\forall \omega \in \Omega$, $\circ_P(\pi)=\dfrac{\pi(\omega)}{h(\pi)}$. Therefore, intuitively to get for instance the upper endpoint of $\circ_c(I\pi)(\omega)$ it is enough to select a compatible distribution that provides the smallest value for $\pi(\omega)$ (namely, when it is possible $\pi(\omega)=\lfloor I\pi(\omega) \rfloor$) and the largest value for $h(\pi)$ (namely, when it is possible $h(\pi) = \lceil h(I\pi) \rceil$).

The following two propositions give these endpoints depending whether there exist a unique interpretation or several interpretations having their upper endpoints equal to $\lceil h(I\pi) \rceil$.

Proposition 2. *Let $I\pi$ be a sub-normalized interval-based possibility distribution. If there exist more than one interpretation having their upper endpoints equal to $\lceil h(I\pi)\rceil$, then $\forall\omega \in \Omega$:*

$$\circ_c(I\pi)(\omega) = \left[\frac{\lfloor I\pi(\omega)\rfloor}{\lceil h(I\pi)\rceil}, min\left(1, \frac{\lceil I\pi(\omega)\rceil}{\lfloor h(I\pi)\rfloor}\right)\right]$$

Proof. Let $\omega\in\Omega$ be an interpretation.

- The lower endpoint $\lfloor\circ_c(I\pi)(\omega)\rfloor$ is equal to $\frac{\lfloor I\pi(\omega)\rfloor}{\lceil h(I\pi)\rceil}$. Indeed, this possibility degree $\frac{\lfloor I\pi(\omega)\rfloor}{\lceil h(I\pi)\rceil}$ exists and is obtained by considering a compatible possibility distribution π where $h(\pi) = \lceil h(I\pi)\rceil$ (remember that $\pi(\omega) \leq h(\pi)$). Besides, since for each compatible possibility distribution π' we have $h(\pi') \leq \lceil h(I\pi')\rceil$ and $\pi'(\omega) \geq \lfloor I\pi(\omega)\rfloor$ then $\circ_P(\pi')(\omega) \geq \frac{\lfloor\pi(\omega)\rfloor}{\lceil h(I\pi)\rceil}$.
- Similarly, the upper endpoint $\lceil\circ_c(I\pi)(\omega)\rceil$ is equal to $min\left(1, \frac{\lceil I\pi(\omega)\rceil}{\lfloor h(I\pi)\rfloor}\right)$. Again, this possibility degree $min\left(1, \frac{\lceil I\pi(\omega)\rceil}{\lfloor h(I\pi)\rfloor}\right)$ exists and is obtained, by considering a compatible possibility π where $h(\pi) = \lfloor h(I\pi)\rfloor$ and $\pi(\omega) = min\left(\lfloor h(I\pi)\rfloor, \lceil I\pi(\omega)\rceil\right)$. Let us show that for every compatible possibility distribution π', we have $\circ_P(\pi')(\omega) \leq min\left(1, \frac{\lceil I\pi(\omega)\rceil}{\lfloor h(I\pi)\rfloor}\right)$. Let us consider two cases:
  - If $\lceil I\pi(\omega)\rceil < \lfloor h(I\pi)\rfloor$ then for every compatible possibility distribution π', we have:
    $$\pi'(\omega) \leq \lceil I\pi(\omega)\rceil \text{ (hence}$$
    $$\pi'(\omega) \leq \min\left(\lfloor h(I\pi)\rfloor, \lceil I\pi(\omega)\rceil\right) \text{ since } \lceil I\pi(\omega)\rceil < \lfloor h(I\pi)\rfloor)$$
    and
    $$h(\pi') \geq \lfloor h(I\pi)\rfloor.$$
    Therefore
    $$\circ_P(\pi')(\omega) = \frac{\pi'(\omega)}{h(\pi')} \leq \frac{min\left(\lfloor h(I\pi)\rfloor, \lceil I\pi(\omega)\rceil\right)}{\lfloor h(I\pi)\rfloor} = \min\left(1, \frac{\lceil I\pi(\omega)\rceil}{\lfloor h(I\pi)\rfloor}\right).$$
  - If $\lceil I\pi(\omega)\rceil \geq \lfloor h(I\pi)\rfloor$ then trivially:
    $$\circ_P(\pi')(\omega) \leq \min\left(1, \frac{\lceil I\pi(\omega)\rceil}{\lfloor h(I\pi)\rfloor}\right)$$
    since $min\left(1, \frac{\lceil I\pi(\omega)\rceil}{\lfloor h(I\pi)\rfloor}\right) = 1$.

□

Next proposition concerns a very particular situation where there exists exactly one interpretation ω such that $\lceil\circ_c(I\pi)(\omega)\rceil=\lceil h(I\pi)\rceil$. In this case, only the lower endpoint of the interpretation ω will differ. More precisely:

Proposition 3. *Let $I\pi$ be a sub-normalized interval-based possibility distribution. If there exists exactly one interpretation ω such that $\lceil I\pi(\omega)\rceil=\lceil h(I\pi)\rceil$. Define $secondbest(I\pi)$=max$\{\lceil I\pi(\omega')\rceil : \omega' \in \Omega$ and $\lceil I\pi(\omega')\rceil \neq \lceil h(I\pi)\rceil\}$. Then:*

$$\circ_c(I\pi)(\omega')=\begin{cases} \left[\frac{\lfloor I\pi(\omega')\rfloor}{\lceil h(I\pi)\rceil}, min\left(1, \frac{\lceil I\pi(\omega')\rceil}{\lfloor h(I\pi)\rfloor}\right)\right] if\ \omega' \neq \omega \\ [1,1] if\ \omega' = \omega\ and\ secondbest(I\pi) = 0 \\ \left[\frac{\lfloor I\pi(\omega)\rfloor}{secondbest(I\pi)}, 1\right] \text{ otherwise.} \end{cases}$$

Proof. In the situation where there exists exactly one interpretation ω such that $\lceil I\pi(\omega)\rceil=\lceil h(I\pi)\rceil$. Then first $\forall\omega'\neq\omega$, we have :

$$\circ_c(I\pi)(\omega') = \left[\frac{\lfloor I\pi(\omega')\lfloor}{\lceil h(I\pi)\rceil}, min\left(1, \frac{\lceil I\pi(\omega')\rceil}{\lfloor h(I\pi)\rfloor}\right)\right]$$

The proof for this case is exactly the same as the one given in Proposition 2.

Now regarding the interpretation ω, there are two cases to consider :

- if $secondbest(I\pi)$=0 then this means that $\forall\omega' \neq \omega, I\pi(\omega')$=$[0,0]$. Hence, for each compatible possibility distribution π we have $\circ_P(\omega)$=1 and $\forall\omega'\neq\omega$, $\circ_P(\omega')$=0. Hence, $\circ_c(I\pi)(\omega)$=$[1,1]$ and $\forall\omega'\neq\omega$, $\circ_c(I\pi)(\omega')$=$[0,0]$.
- if $secondbest(I\pi) \neq 0$ then this means that $\exists\omega'\neq\omega$, such that $I\pi(\omega')\neq[0,0]$. In this case,

$$\circ_c(I\pi)(\omega) = \left[\frac{\lfloor I\pi(\omega)\rfloor}{secondbest(I\pi)}, 1\right].$$

The upper endpoint (1) is obtained by considering a compatible possibility distribution π where $\pi(\omega)$=$\lceil h(I\pi)\rceil$. The lower endpoint is obtained by considering another compatible possibility distribution π' where $\pi'(\omega)$=$\lfloor I\pi(\omega)\rfloor$ (the smallest possible degree for ω in $I\pi$) and for some ω', $\pi'(\omega)$=$secondbest(I\pi)$ (such ω' exists by assumption that $secondbest(I\pi) \neq 0$). One can check that for each compatible possibility distribution π" we have $\pi"(\omega)\geq\lfloor I\pi(\omega)\rfloor$ and $h(\pi") \leq secondbest(I\pi)$. Therefore, $\circ_P(\pi")(\omega)$=$\frac{\pi"(\omega)}{h(\pi")} \geq \frac{\lfloor I\pi(\omega)\rfloor}{secondbest(I\pi)}$.

□

The following algorithm summarizes the computation of $\circ_c(I\pi)$ on the basis of the above propositions.

Algorithm 1 Compatible-based normalization of interval-based possibility distributions

Input: An interval-based possibility distribution $I\pi$
Output: $\circ_c(I\pi)$ a normalized interval-based possibility distribution
  if $\forall \omega \in \Omega, I\pi(\omega) = [0,0]$ (degenerate case) **then**
    $\forall \omega \in \Omega, \circ_c(I\pi)(\omega) = [1,1]$
  else if $I\pi$ is weakly normalized **then**
    $\circ_c(I\pi) = I\pi$
  else if there exists a unique ω such that $\lceil I\pi(\omega)\rceil = \lceil h(I\pi)\rceil$ **then**
    if $\forall \omega' \neq \omega, I\pi(\omega') = [0,0]$ **then**
      $\circ_c(I\pi)(\omega) = [1,1]$
      $\forall \omega' \neq \omega, \circ_c(I\pi)(\omega') = [0,0]$.
    else
      Let $secondbest(I\pi)$=max$\{\lceil I\pi(\omega')\rceil$: $\omega'\in\Omega$ and $I\pi(\omega') \neq \lceil h(I\pi)\rceil\}$.
      $\circ_c(I\pi)(\omega) = [\frac{\lfloor I\pi(\omega)\rfloor}{secondbest(I\pi)}, 1]$
      $\forall \omega'\neq\omega, \circ_c(I\pi)(\omega') = \left[\frac{\lfloor I\pi(\omega')\rfloor}{\lceil h(I\pi)\rceil}, \min\left(1, \frac{\lceil I\pi(\omega')\rceil}{\lfloor h(I\pi)\rfloor}\right)\right]$
    end if
  else
    $\forall \omega \in \Omega, \circ_c(I\pi)(\omega) = \left[\frac{\lfloor I\pi(\omega)\rfloor}{\lceil h(I\pi)\rceil}, \min\left(1, \frac{\lceil I\pi(\omega)\rceil}{\lfloor h(I\pi)\rfloor}\right)\right]$
  end if

The following proposition shows that $\circ_c(I\pi)$ is, in general, weakly normalized. It also provides under which conditions $\circ_c(I\pi)$ is strongly normalized.

Proposition 4. *Let $I\pi$ be an interval-based possibility distribution. Then:*

- $\circ_c(I\pi)$ *is weakly normalized.*
- $\circ_c(I\pi)$ *is strongly normalized if and only if there exists an interpretation ω such that $\forall \omega' \neq \omega$, $\lfloor I\pi(\omega)\rfloor \geq \lceil I\pi(\omega')\rceil$.*

Proof. Let $I\pi$ be an interval-based possibility distribution. Then:

- To see that $\circ_c(I\pi)$ is weakly normalized, it is enough to choose a compatible possibility distribution π and an interpretation ω such that $\pi(\omega)=\lceil h(I\pi)\rceil$. This means that $\forall \omega' \neq \omega$, $\pi(\omega) \geq \pi(\omega')$. Hence, $\circ_P(\pi)(\omega)=1$. Therefore $\lceil \circ_c(I\pi)(\omega)\rceil=1$.
- Let us show that $\circ_c(I\pi)$ is strongly normalized if and only if there exists an interpretation ω such that $\forall \omega'\neq\omega$, $\lfloor I\pi(\omega)\rfloor \geq \lceil I\pi(\omega')\rceil$.
  - Assume that there exists an interpretation ω such that $\forall \omega'\neq\omega$, $\lfloor I\pi(\omega)\rfloor\geq\lceil I\pi(\omega')\rceil$. Then for each compatible possibility distribution π, we have $\forall \omega'\neq\omega$, $\pi(\omega)\geq\pi(\omega')$. This also means that in each compatible possibility distribution π we have $\circ_P(\pi)(\omega) = 1$. Therefore $\lfloor \circ_c(I\pi)(\omega)\rfloor=1$ and hence $\circ_c(I\pi)(\omega) = [1,1]$ and $\circ_c(I\pi)$ is strongly normalized.
  - Now assume there is no interpretation ω such that $\forall \omega'\neq\omega$, $\lfloor I\pi(\omega)\rfloor\geq\lceil I\pi(\omega')\rceil$. Then for each $\omega\in\Omega$, it is possible to get a compatible distribution π such $\pi(\omega)<\pi(\omega')$ for some $\omega'\in\Omega$. Therefore $\lfloor \circ_c(I\pi)(\omega)\rfloor<1$ and hence $\circ_c(I\pi)$ is not strongly normalized.

□

Example 4. *Let us consider again Example 3. From this example, we have $\lfloor h(I\pi)\rfloor$=.5 and $\lceil h(I\pi)\rceil$=.9. Besides, there exists exactly one interpretation such that $\lceil I\pi(\omega)\rceil$=$\lceil h(I\pi)\rceil$; this interpretation is ω_1. Now, $secondbest(I\pi)$=.8. Lastly, the normalization of $I\pi$ of Example 3 gives the distribution of Table 3.*

$\omega_i\in\Omega$	$I\pi(\omega)$
ω_1	[1, 1]
ω_2	$[\frac{.5}{.9}, 1]$
ω_3	[0, 0]
ω_4	[0, 0]

Table 3: Normalized distribution for $I\pi$ of Example 3.

Conclusions

This paper dealt with foundational issues of interval-based possibilistic knowledge bases. More precisely, it dealt with the issue of normalizing an interval-based possibility distribution underlying an interval-based possibilistic base. The normalization is based on the concept of compatible standard possibility distribution. We showed that using the min-based normalization, the result is not an interval-based distribution. However, when the product-based normalization is used, the result is an interval-based possibility distribution. We provided precise lower and upper endpoints of the result of normalizing interval-based possibility distributions. Future works will analyse the syntactic counterpart of the normalization procedure when the input is an inconsistent interval-based possibilistic knowledge base.

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